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Phases of $\mathcal{N} = 1$ supersymmetric gauge theories

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Abstract: $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N)$ gauge theory with adjoint matter Φ and a polynomial superpotential $\mathrm{Tr}\, W(\Phi)$ has been much studied recently. The classical theory has several vacua labeled by integers $(N_1, N_2, \ldots, N_k)$, with the classical unbroken gauge group $\prod_i \mathrm{U}(N_i)$. Quantum mechanically, each classical vacuum leads to $\prod_i N_i$ different vacua. As the parameters of $W(\Phi)$ are varied, these vacua change in a continuous (and holomorphic) fashion. We find that vacua associated with $(N_1, N_2, \ldots, N_k)$ can be continuously transformed to vacua with $(\widetilde{N}_1, \widetilde{N}_2, \ldots, \widetilde{N}_k)$, thus leading to a new kind of duality. Traditional order parameters, like the Wilson loop and 't Hooft loop, sometimes distinguish different phases. We also find phases that are not distinguished by conventional order parameters. The whole picture of the phase diagram is reminiscent of the phase diagram of M-theory.

Keywords: Nonperturbative Effects, Confinement, Supersymmetry and Duality, Matrix Models.

Contents

1. Introduction

Dynamics of four-dimensional supersymmetric gauge theories has proved to be a remarkably rich subject — one that hopefully will ultimately be important for the understanding of nature! Among other attractions of these theories, models with $\mathcal{N} = 1$ supersymmetry exhibit many subtle properties, such as dynamical generation of mass, confinement of charge, and dynamical symmetry breaking, that are seen in the world of strong interactions and are still not fully understood (for a review, see e.g. [1]). One example that has been much-studied in recent years is a $\mathrm{U}(N)$ gauge theory with a chiral superfield Φ in the adjoint representation and a general single-trace superpotential $\mathrm{Tr}\, W(\Phi)$, for some polynomial function W that we will take to be of degree $k+1$ and suitably generic. This model has been studied by its relation to geometric transitions and mirror symmetry in string theory [2, 3, 4] and more recently by its surprising relation [5, 6, 7] to a bosonic matrix model —

a zero-dimensional model with an $\widehat{N}\times\widehat{N}$ matrix M and potential $\mathrm{Tr}\,W(M)$, where $\widehat{N}\to\infty$. In this paper, we will investigate some additional fascinating properties of this example.

Let us first explain the original question that motivated our investigation. In this model, if the function $W'(x)$ has critical points a_i, $i=1,\ldots,k$, then a classical vacuum is chosen by taking Φ to be a diagonal matrix whose eigenvalues are the a_i, taken with multiplicity N_i; the N_i are any non-negative integers such that $\sum_i N_i = N$. We let n be the number of choices of i with $N_i > 0$. Clearly, $n \le k$. At the classical level, the choice of Φ spontaneously breaks the gauge group from $\mathrm{U}(N)$ to $\prod_i \mathrm{U}(N_i)$ (where we include in the product only the positive N_i). Locally, this group is isomorphic to $\mathrm{U}(1)^n \times \prod_i \mathrm{SU}(N_i)$. At low energies, the $\mathrm{SU}(N_i)$ gauge theories become confining and massive, so quantum mechanically, the gauge group that is actually observed at low energies is $\mathrm{U}(1)^n$.

Since this is independent of N_i, the question now arises of whether it is possible quantum mechanically to distinguish in a precise way between vacua that have the same value of n but different N_i. To make this question clear, start with a choice of $W(\Phi)$ such that the a_i are far apart and the classical description of the Φ field is a good approximation, and put the theory in a vacuum characterized by a given choice of integers N_i. Now vary the parameters in $W(\Phi)$, possibly passing through a region in which a_i are not far apart, but then return to a semiclassical region in which the a_i are widely separated. In such a process, can one interpolate from a vacuum with one set of integers N_i to a vacuum with another set of integers $\widetilde{N}_i$?

Let us provide a more systematic framework for this question. For each choice of $W(\Phi)$, the theory under investigation has only a finite set of vacua. We will allow the parameters in W to vary, keeping only the leading coefficient (and the microscopic gauge coupling parameter of the underlying $\mathrm{U}(N)$ theory) fixed. The varying parameters describe a complex manifold $\mathcal{T} \cong \mathbf{C}^k$.[1] As we vary the parameters in $\mathcal{T}$, the vacua fit into a moduli space $\mathcal{M}$ which because of supersymmetry is a complex manifold. $\mathcal{M}$ is a union of components (irreducible complex submanifolds) $\mathcal{M}_\alpha$ which may possibly intersect at singularities. Each component $\mathcal{M}_\alpha$ is a finite cover of $\mathcal{T}$, and hence by going to infinity in $\mathcal{T}$, we reach in each $\mathcal{M}_\alpha$ at least one semiclassical "end" with a definite set of N_i. Our question is now whether the same $\mathcal{M}_\alpha$ can have different ends with different sets of N_i.

To avoid undue suspense, let us assert that the answer to this question has turned out to be "yes." In considering the question, however, it soon becomes clear that one really should ask all the questions that are usually asked in string theory and M-theory. What are the components of $\mathcal{M}$ and what distinguishes them? What semi-classical ends do they have and to what extent do they intersect at singularities? What is the physics at these singularities? (Some examples of singularities have been studied in [8].) When $\mathcal{M}$ does have distinct components $\mathcal{M}_\alpha$, what kind of natural "off-shell" interpolations can one make between them? Without going back to the underlying $\mathrm{U}(N)$ gauge theory (which really gives the most incisive definition of the whole problem), what kind of a low energy phenomenological lagrangian or a "string field theory" can one find that describes all of the $\mathcal{M}_\alpha$ at once?

[1] In this counting, we discard an irrelevant additive constant in $W(\Phi)$ and set the highest power in $W(x)$ to $\frac{g_k}{k+1}x^{k+1} = \frac{1}{k+1}x^{k+1}$.

Smooth interpolations between classical limits with different gauge groups have been found in some other four-dimensional models. For example, in [9], in the context of M-theory compactification on a G_2 manifold, an interpolation from $\mathrm{SO}(8+2n)$ to $\mathrm{Sp}(n)$ was argued, along with several other examples. This was generalized by Friedmann [10], who considered some additional examples, including models with $\mathrm{SU}(N)$ gauge symmetry broken by Wilson lines to $\prod_{i=1}^{n} \mathrm{SU}(N_i) \times \mathrm{U}(1)^{n-1}$, and proposed interpolation between different sets of N_i in that case. Because the symmetry breaking pattern is the same, one may wonder if the model is actually equivalent by some string duality to the one considered in the present paper. One obstacle to finding such a duality is that it is not clear what the superpotential $W(\Phi)$ would correspond to in the model considered in [10]. At any rate, the present paper is devoted to studying a field theory problem by field theory methods; our interpolations and dualities are purely field theoretic and do not depend on stringy phenomena.

It will become clear that the subject we are exploring is very rich, and in the present paper we will really only describe some general properties and explore some examples. Most likely, many interesting phenomena remain to be uncovered.

Since it turns out that $\mathcal{M}$ does have many components, one basic question is how to distinguish them. One fundamental invariant that we have already identified is the rank n of the true low energy gauge group. In many physical theories, another important invariant comes from the realization of symmetries: which subgroup of the global symmetry group is unbroken in a given component of the moduli space of vacua? In the present paper, this criterion will be somewhat less useful, because for generic W there generally are no global symmetries; still, the realization of global symmetries, and more generally, transformations of Φ that must also be taken to act on the parameters in $\mathcal{T}$, will give some important information. Additional important information will come from a more subtle but still standard type of order parameter that is provided by confinement: for an external Wilson loop in a given representation of $\mathrm{U}(N)$, does one observe a Coulomb law or an area law?

All these standard order parameters will play a major role in the present paper, but we will see that they do not suffice. We also will encounter a more subtle sort of order parameter that only can be defined using supersymmetry. It may be characterized as follows. Suppose that a theory with $\mathcal{N}=1$ supersymmetry has r independent chiral superfields $\mathcal{O}_s$, $s=1,\ldots,r$.[2] By supersymmetry, the $\mathcal{O}_s$ (or more precisely their expectation values) are holomorphic functions on $\mathcal{M}$. Now suppose that r exceeds the complex dimension of the moduli space $\mathcal{M}$. Then as functions on any given component $\mathcal{M}_\alpha$, the $\mathcal{O}_s$ inevitably obey some algebraic equations; if these equations are different for different $\mathcal{M}_\alpha$, this gives an order parameter of sorts by which the $\mathcal{M}_\alpha$ can be distinguished. We will find many examples of this situation, where the $\mathcal{M}_\alpha$ cannot be distinguished by any conventional order parameter that we can see but can be distinguished by the chiral equations that are obeyed on them. When this occurs, we do not know if the α-dependence of these chiral equations should be regarded as an explanation for the existence of different branches, or merely as a description of the phenomenon.

[2] When we say that they are independent, we mean that they obey no relations in the chiral ring of the theory that are independent of the parameters g_k in the superpotential. Since we are working over the parameter space $\mathcal{T}$, we are treating the g_k as variables rather than as fixed complex numbers.

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Organization of the paper. This paper is organized as follows. Section 2 is devoted to general considerations. In section 2.1, we describe what can be predicted about the moduli space of vacua from a "weak coupling" (actually weak gauge coupling) point of view in which the effects of the superpotential $W(\Phi)$ are assumed to set in at much higher energies than the effects of the gauge coupling. We focus on understanding what can be said about the moduli space of vacua using the expected confinement of the low energy gauge theory.

In section 2.2, following previous literature [4, 11], we consider a strong (gauge) coupling point of view, in which the superpotential is considered as a small perturbation of a strongly coupled gauge theory with $W = 0$; this theory has $\mathcal{N} = 2$ supersymmetry and a vacuum structure determined in [12, 13, 14]. In previous work, the strong coupling point of view was used to analyze the vacua with a given rank n of the unbroken symmetry group. That analysis assumed that the N_i defined above are all nonzero and that the degree of W is not too large. We generalize the analysis to remove these restrictions. (This analysis is completed in appendix A.)

In section 2.3, we review a construction originally described in [4] whereby, given vacua for the $\mathrm{U}(N)$ gauge theory with any given set of N_i, and any integer $t > 1$, one can construct vacua for the $\mathrm{U}(tN)$ gauge theory with $N_i' = tN_i$. Comparing to our analysis of confinement in section 2.1, we will see that the vacua arising this way are precisely the confining vacua in which the confinement index (defined in section 2.1) is t. Conversely, all vacua with confinement index t arise by this construction. Since the theory with $N_i' = tN_i$ also has vacua without confinement or with lower confinement index (as will be clear in section 2.1), there must be additional vacua with $N_i' = tN_i$; they will be studied (in examples) in section 3.

In section 2.4, we present some ways in which different branches $\mathcal{M}_\alpha$ can arise and some constraints on the possible classical limits of a given branch. We also describe the simplest mechanism by which different branches may intersection.

In section 3, we will explore the question of interpolating from one set of N_i to another. For this, we want to consider the simplest case in which n is as small as possible. For $n = 1$, the gauge group is completely unbroken at the classical level, so the question of interpolating between classical limits with different unbroken gauge groups does not arise. So the first case is really that the rank of the low energy gauge group is $n = 2$ which (in the minimal case with all N_i non-vanishing) arises for a cubic superpotential, $k = 2$. Since we do not know how to make the analysis for general N, we will explore in detail the cases of $N = 2, 3, 4, 5$, and 6. For these values of N and $k = n = 2$, we determine in detail all branches of the moduli space of vacua, recovering the confining vacua that were found in section 2.3 and describing the vacua without confinement. For the vacua without confinement, we do get the promised smooth continuations between different values of N_i, including interpolation from $(N_1, N_2) = (1, 3)$ to $(2, 2)$ for $N = 4$, from $(1, 4)$ to $(2, 3)$ for $N = 5$, and from $(1, 5)$ to $(2, 4)$ to $(3, 3)$ for $N = 6$. We conjecture that for any N, smooth interpolations are possible on the Coulomb branch from any (N_1, N_2) to any other $(\widetilde{N}_1, \widetilde{N}_2)$.

In this detailed description for $2 \leq N \leq 6$, we also see how branches with $n = 2$ meet branches with $n = 1$ at singularities. In addition to these intersections at finite points on the moduli space, we find for $2 \leq N \leq 6$ that all branches meet the Coulomb branch at infinity, where the semiclassical description is valid.

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In section 4, we consider the opposite limit of large-N. For $n = N$, that is all $N_i = 1$, there is a unique vacuum for each function $W(\Phi)$ and so our question about the different branches does not arise. So we consider the case that $N - n$ is small but positive. Here, the simple strong coupling analysis of section 2.2 predicts that there must be at least $N - n + 1$ branches of vacua. As far as we know, they are not distinguished by conventional order parameters. We show, as suggested above, that they can be distinguished by determining which holomorphic functions of the chiral superfields vanish on each branch. We do not know how best to interpret this result.

In two appendices, we give more details of the strong gauge coupling analysis. In appendix A, we extend the proof that relates the matrix model curve to the $\mathcal{N} = 2$ curve to W of high degree. In appendix B, we show how the generalized Konishi anomaly [15, 16] which was derived and used in [17] arises in the strong gauge coupling regime where the elementary gauge fields are not visible. In appendix C we list some useful properties of Chebyshev polynomials. Appendix D defines a magnetic index ν which characterizes the confinement in a theory with massless photons. It is related to the confinement index t of section 2.3 through $N = t\nu$.

2. General considerations

2.1 Confinement

Here, we will see what we can say about the U(N) gauge theory with adjoint superfield Φ by using confinement as an order parameter. First we consider the matter directly in field theory, and then we re-examine the issues using the matrix model.

We consider a classical vacuum in which N_i eigenvalues of Φ are placed at the i^{th} critical point of the superpotential W. To keep the exposition simple, we will assume that the N_i are all positive. We also assume that the critical points of W are distinct, so that all components of Φ are massive classically. Supposing that the underlying U(N) gauge theory is weakly coupled at the scale set by those masses, the low energy physics is simply that of the pure supersymmetric gauge theory with classical gauge group $G_{low}^{cl} = \mathrm{U}(1)^n \times \prod_{i=1}^n \mathrm{SU}(N_i)$.

If $N_i > 1$, each of the SU(N_i) theories becomes confining at exponentially smaller energies. For any $N_i \geq 1$, the SU(N_i) theory has N_i vacua, each with a mass gap. The total number of vacua for fixed W and fixed N_i is hence $\prod_{i=1}^n N_i$. If only W and G_{low}^{cl} are specified, and not the individual number N_i of eigenvalues at the i^{th} critical point, then the total number of vacua is larger, as one must allow for permutations of eigenvalues among the critical points.

In the limit of small gauge coupling, the different SU(N_i) factors decouple and each of them confines. This confinement can be diagnosed by the possible area law of the Wilson loops W_i of the various SU(N_i). In the full theory, these W_i are not well-defined. Correspondingly, the full SU(N) theory is not necessarily confining even if the sub-theories are. (In this discussion, the U(1) factor in the underlying gauge group $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)$ is unimportant, as it decouples and does not contribute to the dynamics.) For a precise criterion for confinement, we have to use Wilson lines of the full SU(N) theory.

To probe confinement, we consider a large Wilson loop in some representation $\mathcal{R}$ of $\mathrm{SU}(N)$, and ask if it exhibits an area law or a Coulomb law. All that matters about $\mathcal{R}$ is how it transforms under the center of $\mathrm{SU}(N)$, which is $\mathbf{Z}_N$, generated by $\omega = \mathrm{diag}(e^{2\pi i/N}, e^{2\pi i/N}, \ldots, e^{2\pi i/N})$. The reason that only the center matters is that if $\mathcal{R}$ and $\mathcal{R}'$ are two irreducible representations that transform in the same way under the center, then an external charge in the representation $\mathcal{R}$ can combine with gluons to make an external charge in the representation $\mathcal{R}'$, so one of these external charges is confined (has infinite energy) if and only if the other is. Turning one representation into another by combining with gluons is a process that we may call electric screening.

To give an example with any desired action of the center, we can simply take $\mathcal{R}$ to be the tensor product of r copies of the fundamental representation, for some $r \geq 0$. If W is a Wilson loop in the fundamental representation, the Wilson loop for the representation $\mathcal{R}$ is just W^r. Only the value of r modulo N matters.

If two representations $\mathcal{R}$ and $\mathcal{R}'$ are unconfined, then so is the tensor product $\mathcal{R} \otimes \mathcal{R}'$. So the set of values of r for which there is no confinement is closed under addition. There is no area law for $r = N$ (since $r = N$ is equivalent to $r = 0$), so the smallest positive value of r for which there is no area law is always a divisor of N. We denote this number as t and call it the confinement index. If the theory is completely confining (all Wilson loops with non-trivial action of the center have an area law), then $t = N$, and if the theory is completely unconfining (no Wilson loop shows an area law), then $t = 1$.

Let us now give a few examples of the behavior of the $\mathrm{SU}(N)$ theory. Suppose that $\mathrm{SU}(N)$ is broken classically to $\mathrm{SU}(N-1) \times \mathrm{U}(1)$. All vacua will be unconfining simply because there is no confinement in $\mathrm{U}(1)$. The fundamental representation of $\mathrm{SU}(N)$ has a component which is an $\mathrm{SU}(N-1)$ singlet, and this component feels no confining gauge forces. When we evaluate the expectation value of a Wilson line in the fundamental representation $\mathcal{R}$,

$$\left\langle \operatorname*{Tr}_{\mathcal{R}} P \exp \oint_C A \right\rangle, \tag{2.1}$$

the gauge field is effectively an $(N-1) \times (N-1)$ matrix and a commuting 1×1 component associated with the $\mathrm{U}(1)$ factor (plus massive fields that will not influence the question of confinement). In computing the expectation value in (2.1), assuming that C is very large compared to the scale at which $\mathrm{SU}(N-1)$ becomes strongly coupled, the dominant contribution will come from the "bottom" component of $\mathcal{R}$. This component does not interact with the $\mathrm{SU}(N-1)$ gauge fields, so there is no area law. Since this component interacts with the massless $\mathrm{U}(1)$ gauge field, the Wilson loop (2.1) exhibits the behavior of a Coulomb phase rather than a Higgs phase.

Computation of confinement index in an example. For a simple example with a non-trivial confinement index, we will explore the case that $\mathrm{SU}(N)$ is broken to $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2) \times \mathrm{U}(1)$. The fundamental representation of $\mathrm{SU}(N)$ decomposes as $(\mathbf{N_1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{N_2})$ under $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$. (We ignore the $\mathrm{U}(1)$ charges, which will not affect confinement.) The tensor product of N_1 copies of $(\mathbf{N_1}, \mathbf{1})$ contains a singlet. So, as in the above example, the Wilson line W^{N_1} will show no area law. Likewise, W^{N_2} will show no area law.

Let t' be the greatest common divisor of N_1 and N_2. Absence of confinement for $r = N_1$ and for $r = N_2$ implies that there is no confinement for $r = l_1 N_1 + l_2 N_2$ for any positive integers l_1, l_2. We can pick l_1 and l_2 so that $l_1 N_1 + l_2 N_2$ is congruent mod N to t'. So the confinement index t can be no bigger than t', and if electric screening, which we have considered so far, were the only mechanism, it would equal t' precisely.

However, there is another mechanism, magnetic screening. Let us first recall 't Hooft's description of confined phases of SU(N) gauge theory. One introduces an 't Hooft loop, which we will call H; H is constructed using a twist by an element of the center of SU(N). The general loop order parameter is $W^r H^s$, where r, s both take values from 0 to $N-1$ (alternatively, their values only matter modulo N). Massive phases are described by saying for which values of r, s there is no area law; one says that charges with this value of r and s are "condensed." (In some situations, for example a confining theory obtained by a small perturbation of an $\mathcal{N} = 2$ theory, one can make the intuition behind this language precise [12].) In particular, there are N possible confining phases; in the r^{th} such phase, $W^r H$ has no area law, or equivalently a charge with quantum numbers $W^r H$ is "condensed."

$\mathcal{N} = 1$ super Yang-Mills theory with gauge group SU(N) has N confining vacua, with each type appearing precisely once. In fact, an adiabatic increase of the theta angle by 2π has the effect of increasing r by 1.

So when we break the underlying SU(N) theory to $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2) \times \mathrm{U}(1)$, we get a low energy theory that indeed has $N_1 N_2$ vacua, but they are not equivalent. They can be distinguished by the type of confinement. For each pair r_1, r_2 (with $0 \leq r_i \leq N_i - 1$), there is one vacuum in which $W_1^{r_1} H_1$ and $W_2^{r_2} H_2$ are unconfined. (Here we write W_1 and W_2 for Wilson lines in the fundamental representations of SU(N_1) and SU(N_2), respectively, and similarly H_1 and H_2 are the 't Hooft loops of the two factors of the low energy gauge group. As we remarked above, this makes sense only near the weak gauge coupling limit.)

This theory also has 't Hooft-Polyakov magnetic monopoles. If $N = 2$, $N_1 = N_2 = 1$, we take the basic SU(2) monopole whose magnetic field at infinity is (in a unitary gauge) a multiple of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

For any N_1, N_2, and N, we just embed the SU(2) monopole in SU(N) in such a way that it does not fit entirely in either SU(N_1) or SU(N_2) — we take the "1" in the upper left to act in SU(N_1) and the "-1" in the lower right to act in SU(N_2). (We could also embed the monopole entirely in SU(N_1) or entirely in SU(N_2), but this would give nothing interesting for our present purposes.)

Now suppose we probe the (r_1, r_2) vacuum of the full SU(N) theory with the external Wilson loop $W^{r_1 - r_2}$. Although $r_1 - r_2$ is not necessarily a multiple of t', we claim that this Wilson loop will not show an area law. The external SU(N) charge $r_1 - r_2$ can, after symmetry breaking to $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$, be divided between the two factors in various ways, and confinement will be avoided if there is any way to divide the charge to make the energy finite. We simply put r_1 units of charge in SU(N_1) and $-r_2$ units in SU(N_2). (The charges are only well-defined modulo N_1 and N_2.) So in other words, $W^{r_1 - r_2}$ can behave in the low energy theory as $\widetilde{W} = W_1^{r_1} W_2^{-r_2}$. But the external Wilson line $\widetilde{W}$ can be screened by the spontaneous nucleation from the vacuum of the magnetic monopole described in the last paragraph. Since the monopole has magnetic charge 1 in $\mathrm{SU}(N_1)$ and -1 in $\mathrm{SU}(N_2)$, it can be represented as $H_1 H_2^{-1}$. So in conjunction with a monopole, $\widetilde{W}$ can be represented in the low energy theory as $W_1^{r_1} H_1 W_2^{-r_2} H_2^{-1}$, and has no area law since (by the definition of r_1 and r_2), $W_1^{r_1} H_1$ has no area law in $\mathrm{SU}(N_1)$, and $W_2^{-r_2} H_2^{-1}$ (which is obtained from $W_2^{r_2} H_2$ by reversing the orientation of the loop) has no area law in $\mathrm{SU}(N_2)$.

The net effect is that there is no area law for W^{N_1}, for W^{N_2}, or for $W^{r_1 - r_2}$. Hence the confinement index t is at most the greatest common divisor of N_1, N_2, and $b = r_1 - r_2$. We claim that this is the correct value, since by now we have considered screening by all of the electric and magnetic objects that exist in the theory.

It is easy to see why the confinement index depends not on r_1 and r_2 separately but only on their difference b. In fact, under $\theta \to \theta + 2\pi$, one has $r_i \to r_i + 1$, for $i = 1, 2$, while external Wilson loops are unaffected. The confinement index must be invariant under this operation, so can only depend on the difference of the r_i.

General case. We can easily extend this analysis to the general case that $\mathrm{U}(N)$ is broken down to $\prod_{i=1}^n \mathrm{U}(N_i)$. Electric screening ensures that there is no area law for W^{N_i}, $i = 1, \ldots, n$. In addition, we must consider magnetic screening. The low energy theory has $\prod_{i=1}^n N_i$ vacua, which are characterized by giving integers r_i, $i = 1, \ldots, n$ (with $0 \leq r_i \leq N_i - 1$) such that in the low energy $\mathrm{SU}(N_i)$ theory, $W_i^{r_i} H_i$ has no area law. A magnetic monopole can be shared between $\mathrm{SU}(N_i)$ and $\mathrm{SU}(N_{i+1})$ in such a way that it has charges $H_i H_{i+1}^{-1}$.[3] It follows, just as in the case with $n = 2$ that was considered above, that the external Wilson line $W^{r_i - r_{i+1}}$ has no area law. (To make the argument, we simply place r_i units of charge in $\mathrm{SU}(N_i)$ and $-r_{i+1}$ units in $\mathrm{SU}(N_{i+1})$ and then screen this configuration using the monopole.)

So if we let $b_i = r_i - r_{i+1}$, the confinement index t is now the greatest common divisor of the N_i and the b_i.

There are t different types of branch with confinement index t. For a branch of the u^{th} type, the operators that do not have an area law are W^t and $W^u H$.

Interpretation in the matrix model. The matrix model is described [4] by a complex curve Σ:

$$y_m^2 = W'(x)^2 + f_{n-1}(x) \,, \tag{2.3}$$

where f depends on the gauge coupling and the N_i. We assume for simplicity that $k = n$; i.e. no N_i vanishes. Σ is a double cover of the x-plane. With W' being of degree n, the right hand side of (2.3) has $2n$ zeroes; the projection of Σ to the complex x-plane is branched at these $2n$ points.

The situation can be described particularly simply for weak coupling, where f is small. The polynomial W' has n roots a_i, and for small f, the polynomial $(W')^2 + f$ has, for each

[3] One can more generally share a monopole between $\mathrm{SU}(N_i)$ and $\mathrm{SU}(N_j)$ for any i, j, but this monopole leads to nothing new because its charges are a linear combination of the charges of monopoles considered in the text.

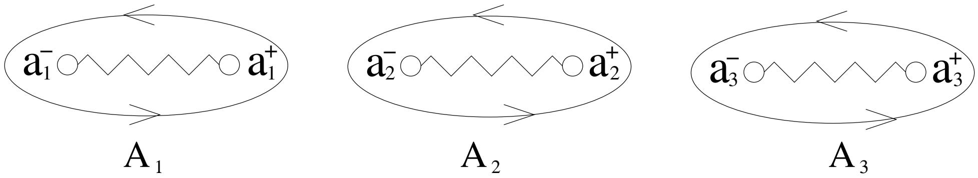

Figure 1: Branch cuts, depicted by zigzags connecting the $a_i^{\pm}$, and cycles A_i surrounding the cuts, are depicted here for $n = 3$.

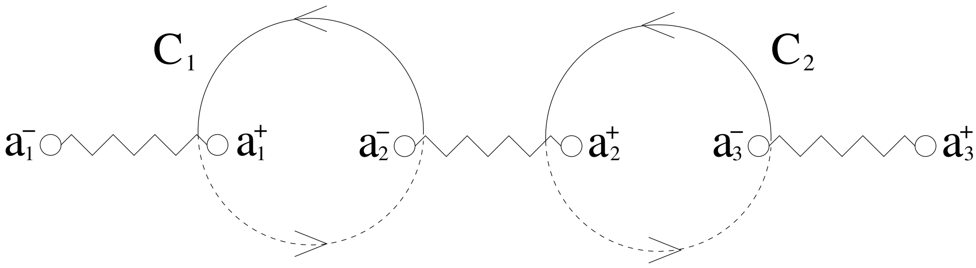

Figure 2: Choice of compact cycles C_i for $n = 3$. The part of the contour depicted by a dashed line is on the second sheet below the cut.

i, a pair of roots $a_i^{\pm}$ that are near a_i.[4] We connect the $a_i^{\pm}$ with branch cuts, as in figure 1, so that y_m is a single-valued function away from these cuts.

In [17], the operator-valued differential

$$T(x) = \mathrm{Tr}\,\frac{dx}{x - \Phi} \tag{2.4}$$

was introduced. It was shown that over a cycle A_i surrounding the i^{th} cut (also shown in figure 1), the period of $T(x)$ was

$$N_i = \oint_{A_i} T\,. \tag{2.5}$$

(To keep the formulas simple, in this paper we define the contour integral $\oint$ to include a factor of $1/2\pi i$.) In the present paper, we want to go away from the weak coupling limit, so we cannot assume that the $2n$ zeroes of $(W')^2 + f$ are neatly paired up. Likewise, we will not have a distinguished set of A_i cycles. So in developing the theory, we will have to include, along with the A_i, the other compact cycles with which the A_i will mix. There are $n-1$ of these, sketched in figure 2. The new cycles, which we call C_i, roughly connect a_i^+ to a_{i+1}^-, for $i = 1, \ldots, n-1$. We define

$$c_i = \oint_{C_i} T\,. \tag{2.6}$$

One might guess that since the N_i are integers, and will mix with the c_i under strong coupling monodromies, the c_i are also integers. We will prove this at the end of this section, using results from [17]. It turns out that the periods of T (2.6) are the integers c_i

[4] When the parameters are such that the roots $a_i^{\pm}$ are real, we ask that $a_i^+ > a_i^-$. More generally, the symbols $\pm$ are just labels.

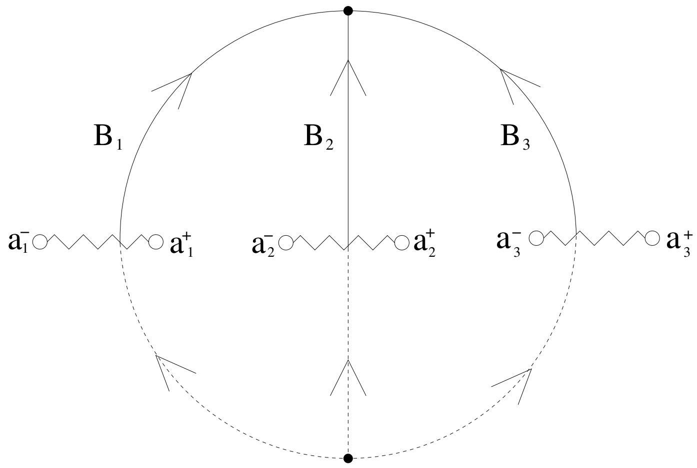

Figure 3: Non-compact cycles B_i for $n = 3$. The bullets represent the two punctures of the Riemann surface (that is, the two points lying above $x = \infty$). The cycle B_i passes through the i^{th} cut and connects the two punctures, which are on different sheets.

only on shell; i.e. only after all the equations of motion have been solved. Off shell, before solving the equations of motion of S_i, the periods around the A_i cycles (2.5) are given by the integers N_i but the periods around the C_i cycles are not constrained. A more detailed discussion of this point will appear below and in a separate publication.

Now let us recall the matrix model construction of the gauge theory effective superpotential. Here one uses another differential, which in the matrix model is

$$R(x) = \frac{g_m}{\widehat{N}} \operatorname{Tr} \frac{dx}{x - M} \tag{2.7}$$

where g_m is the finite 't Hooft coupling of the matrix model. (2.7) has been interpreted in gauge theory in [17]. Its periods are

$$\begin{aligned} \oint_{A_i} R &= S_i \\ \oint_{B_i} R &= \Pi_i = \frac{1}{2\pi i} \frac{\partial \mathcal{F}}{\partial S_i}\,. \end{aligned} \tag{2.8}$$

Here B_i are the noncompact cycles sketched in figure 3. The A_i and B_i are a canonical basis of the homology of the compact surface made by adding points at infinity to Σ; their intersection pairings are $A_i \cap A_j = B_i \cap B_j = 0$, $A_i \cap B_j = \delta_{ij}$. The B_i are related to the compact cycles C_i introduced in the last paragraph by

$$C_i = B_{i+1} - B_i\,. \tag{2.9}$$

Furthermore, $S_i = -\frac{1}{32\pi^2} \operatorname{Tr} W_{\alpha i} W^{\alpha i}$ is the gluino bilinear of $\mathrm{U}(N_i)$ (a more rigorous definition of S_i is given in [17]), and $\mathcal{F}$ is called the "prepotential."

To obtain the matrix model expression for the superpotential, one starts [4] with the fact that the model is dual to string theory on a certain non-compact Calabi-Yau threefold

X. On the threefold, the superpotential is $\frac{1}{2\pi i}\int_X H\wedge\Omega$, where Ω is the covariantly constant holomorphic three-form of X, and the flux H has integral periods on compact cycles. By integrating out the extra dimensions, one reduces this expression to an integral over a Riemann surface (subsequently understood as the matrix model curve). In this reduction, H descends to a differential form $\overline{T}_0$ whose periods on compact cycles are integers, and Ω descends to the differential R. The superpotential becomes $W=\frac{1}{2\pi i}\int_\Sigma \overline{T}_0\wedge R$.[5] Evaluating this in the usual way in terms of contour integrals over one-cycles A_i and B_i (and remembering our factor of $1/2\pi i$ in the definition of a contour integral), we get the usual matrix model formula for the effective superpotential for the S_i:

$$W_{\rm eff}(S_i)=2\pi i\sum_i\left(\oint_{A_i}\overline{T}_0\oint_{B_i}R-\oint_{B_i}\overline{T}_0\oint_{A_i}R\right). \tag{2.10}$$

We will modify previous treatments only in not assuming that $\oint_{B_i}\overline{T}_0$ is independent of i.

According to the familiar analyses [4] of the superpotential, $\oint_{A_i}\overline{T}_0=N_i$. In addition, we set $\oint_{C_i}\overline{T}_0=c_i'$. Let us compare these to the periods of T as given in equations (2.5) and (2.6). Since $\overline{T}_0$ and T_0 have the same periods on the A-cycles, and the A-cycles will mix with the C-cycles under strong coupling monodromies, it is natural to conjecture that also their periods on the C-cycles are equal, $c_i=c_i'$. As will be explained in detail in a separate publication, this holds on-shell, that is after imposing the equations of motion of the S_i. (Off-shell, the C_i periods of T depend on the S_i, while $\overline{T}_0$ has integral periods even before imposing the S_i equations of motion.)

The integrals of $\overline{T}_0$ over the noncompact cycles B_i must then be as follows. $\oint_{B_1}\overline{T}_0=-\tau_0$ for some complex number τ_0, and more generally

$$\oint_{B_i}\overline{T}_0=-\tau_0-b_i\,, \tag{2.11}$$

with

$$b_i=-\sum_{j=1}^{i-1}c_j\,. \tag{2.12}$$

So we get for the effective superpotential

$$W_{\rm eff}(S_i)=\sum_{i=1}^n N_i\frac{\partial\mathcal{F}}{\partial S_i}+2\pi i\tau_0\sum_{i=1}^n S_i+2\pi i\sum_{i=2}^n b_iS_i\,. \tag{2.13}$$

The derivation of (2.13) does not require the knowledge that $\overline{T}_0=T$ on shell.

The only addition that we are making to previous studies of this problem is to include the last term, proportional to the b_i. Let us explain what this term does. The Veneziano-Yankielowicz superpotential for SU(M) supersymmetric gluodynamics is [18]

$$W(S)=S\left[\log\left(\frac{\Lambda^{3M}}{S^M}\right)+M\right]. \tag{2.14}$$

[5] $\overline{T}_0$, like H, is real; we have denoted it $\overline{T}_0$ because R is a differential of type $(1,0)$, so only the $(0,1)$ part of $\overline{T}_0$ actually contributes to the integral.

The logarithm is only defined modulo $2\pi i$, so W is not defined on the S-plane but on an infinite cover thereof where for the time being we allow all possible branches of the logarithm. On this infinite cover, the number of critical points of W is M; the equation for a critical point is indeed

$$\log\left(\frac{\Lambda^{3M}}{S^M}\right) = 0\,, \tag{2.15}$$

and this equation implies that $S^M = \Lambda^{3M}$, an equation that has the familiar M roots. (Moreover, after picking S so $S^M = \Lambda^{3M}$, we must, to obey (2.15), pick the correct branch of the logarithm, so each solution of $S^M = \Lambda^{3M}$ leads to only one vacuum.) Now consider, instead of (2.14), the superpotential

$$\widetilde{W}(S) = M\left[S\log\left(\frac{\Lambda^3}{S}\right) + S\right]. \tag{2.16}$$

Again, this superpotential is defined on an infinite cover of the S plane, but now on this infinite cover, the equation for a critical point is $M\log(\Lambda^3/S) = 0$, which implies $\log(\Lambda^3/S) = 0$. There is only one root, namely $S = \Lambda^3$.

What has happened? Mathematically, there are really M different ways to define an infinite cover of the S-plane on which the Veneziano-Yankielowicz superpotential (2.14) is defined. The reason is that when the argument of S shifts by 2π, $\log(\Lambda^{3M}/S^M)$ shifts by $-2\pi iM$. We do not need to allow all branches of the logarithm to get a space on which W is defined; it suffices to consider $1/M$ of all of the branches. We could pick an arbitrary integer b and say that when Λ^3/S is positive, we want $\log(\Lambda^{3M}/S^M)$ to have an imaginary part congruent to $2\pi ib \pmod{2\pi iM}$. So the Veneziano-Yankielowicz superpotential can be defined on each of M different spaces $\mathcal{S}_b$, $b = 0,\ldots,M-1$, each of which is an infinite cover of the S-plane. Chiral symmetry permutes these, so physically we should include all of them, which is equivalent to including all branches of $\log(\Lambda^{3M}/S^M)$.

The superpotential $\widetilde{W}(S)$ is equivalent to $W(S)$ defined only on $\mathcal{S}_0$. It therefore only describes one of the vacua. To describe the others, in this language, we should explicitly include additional branches on which the superpotential is

$$\widetilde{W}_b = M\left[S\log\left(\frac{\Lambda^3}{S}\right) + S\right] + 2\pi ibS\,. \tag{2.17}$$

The physical meaning of b is very simple. A superpotential term which is an imaginary multiple of S is just a shift in the theta angle. So including b just shifts the theta angle by $2\pi b$, which will rotate confinement (condensation of an 't Hooft loop) to oblique confinement (condensation of a mixture of 't Hooft and Wilson loops). Thus, for given b, the vacuum that is realized has condensation of a b-dependent combination of 't Hooft and Wilson loops W^bH.

Since in this paper we will be interested in studying $\mathrm{U}(M)$ gauge theories rather than $\mathrm{SU}(M)$, we would like to extend (2.17) to this case. As in [17], we define $S = \widehat{S} - \frac{1}{2M}(w_\alpha)^2$. Here $\widehat{S}$ is the trace over the $\mathrm{SU}(M)$ fields and it is taken to be an independent chiral superfield in the effective theory. w_α is the field strength superfield of the $\mathrm{U}(1) \subset \mathrm{U}(M)$.

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A fact which was useful in [17] is the decoupling of this U(1). This decoupling is implemented by considering the "superfield" $\mathcal{S} = S + \psi^\alpha w_\alpha - \psi^1\psi^2 M$, where ψ^α is an auxiliary anticommuting Lorentz spinor. The first term in (2.17) is then written as $\int d^2\psi \left[\frac{1}{2}\mathcal{S}^2 \log(\Lambda^3/\mathcal{S}) + \frac{3}{4}\mathcal{S}^2\right] = M\left[\widehat{S}\log(\Lambda^3/\widehat{S}) + \widehat{S}\right]$, which is independent of w_α. (In fact, any expression of the form $\int d^2\psi H(\mathcal{S})$ is independent of w_α.) The $2\pi i b S$ term in (2.17) can also be written equally well in terms of S in (in U(M)) or $\widehat{S}$ (in SU(M)), since b is an integer and abelian gauge theory is invariant under a 2π shift in the theta angle.

Now let us go back to the matrix model superpotential (2.13), replacing M in the above discussion with any of the N_i. For small S_i, $\partial\mathcal{F}/\partial S_i \approx S_i\left[\log(\Lambda_i^3/S_i) + 1\right]$. So our effective superpotential for small S_i is (we drop the terms which involve two derivatives of the prepotential $\mathcal{F}$)

$$W_{\text{eff}}(S_i) = \sum_i \left(2\pi i \tau_0 S_i + N_i S_i \left[\log\left(\frac{\Lambda_i^3}{S_i}\right) + 1\right] + 2\pi i b_i S_i\right) + \mathcal{O}(S_i S_j)\,. \tag{2.18}$$

The first term is the bare coupling. The second term includes the one loop renormalization of the coupling constant and the strong SU(N_i) IR dynamics. The last term $\mathcal{O}(S_iS_j)$ represents the perturbative contribution of the high energy theory. Our new term $2\pi i b_i S_i$ arises also from the strong IR dynamics of SU(N_i). It should be independent of the photon of $\mathrm{U}(1) \subset \mathrm{U}(N_i)$ and therefore the constants b_i are quantized.

From (2.18) it is clear that the integers b_i represent relative shifts by $2\pi b_i$ of the theta angles of the various SU(N_i). More explicitly, relative to the theta angle in SU(N_1), the theta angle in SU(N_k) is greater by $2\pi b_k$. So if SU(N_1) has condensation of $W^{r_1}H$,[6] then SU(N_k) has condensation of $W^{r_k}H$, where $r_k - r_1 = b_k$. (Our definitions are such that $b_1 = 0$.)

Thus, the b_k that we have defined in (2.12) are the same as the ones we introduced in our general discussion of confinement. So the confinement index t is the greatest common divisor of the N_i and the b_k, or equivalently, the greatest common divisor of the N_i and c_i. Since N_i and c_i are a complete set of periods of T (integrated over compact cycles), we can describe this by saying that the confinement index is the greatest integer t such that all compact periods of T/t are integral. This gives, in the context of the matrix model, a manifestly "modular invariant" definition of the confinement index.

In section 2.3, we will (following [4]) describe an operation that multiplies N by an arbitrary positive integer t, also multiplying the individual N_i by t and multiplying T by t. This operation only generates confining vacua. In fact, it multiplies the confinement index by t. We will see that all confining vacua arise by applying this operation, starting with a non-confining vacuum with a smaller value of N. So for any given N, the "new" vacua that cannot be predicted based on what happened for smaller N are the "Coulomb" vacua, the ones without confinement. For every N, and every set of N_i, there are Coulomb vacua — for example, those vacua in which any of the b_j is 1.

[6]The value of r_1 depends on the bare coupling τ_0 and on how H is defined.

Finally, we explain how to deduce from the results of [17] that the c_i are integers. Instead of studying the present problem using the matrix model curve, we can use the $\mathcal{N}=2$ curve

$$y^2 = P_N(x)^2 - 4\Lambda^{2N}\,. \tag{2.19}$$

(According to [4], these curves are closely related, with $P_N^2 - 4\Lambda^{2N} = H_{N-n}^2((W')^2 + f)$; we will not need here to explicitly use this relation.) In appendix A of [17], it was argued that $T = (P_N'/y)dx$. This can alternatively be written

$$T = \frac{1}{P_N + y} d(P_N + y) = d\log(P_N + y)\,. \tag{2.20}$$

Like any logarithmic derivative, T has integer periods (recall that we include a factor of $\frac{1}{2\pi i}$ in the definition of $\oint$). In other words, $\oint T$ over a compact cycle is the change in $\frac{1}{2\pi i}\log(P_N + y)$ around that cycle, and this is an integer.

2.2 Considerations based on strong gauge coupling

The opposite of the weak gauge coupling approach of section 2.1 is a strong gauge coupling approach in which the superpotential is regarded as a small perturbation of the $\mathcal{N}=2$ gauge dynamics that one would have if $W=0$. This approach was developed in [4] by using methods of [19]. Here we will review this analysis and extend it to allow some of the N_i to vanish and to allow superpotentials of any degree.

We write the superpotential as

$$W = \sum_{r=0}^{k} \frac{g_r}{r+1} \mathrm{Tr}\ \Phi^{r+1}\,. \tag{2.21}$$

In our present discussion, the superpotential is considered a small perturbation of an $\mathcal{N}=2$ $\mathrm{U}(N)$ theory. We interpret W as an effective superpotential on the $\mathcal{N}=2$ Coulomb moduli space by replacing $\mathrm{Tr}\ \Phi^r$ by its vev $\langle \mathrm{Tr}\ \Phi^r\rangle$, regarded as a function on the moduli space. In order to look for vacua in which the low energy gauge group is $\mathrm{U}(1)^n$, we must extremize the superpotential (2.21) constrained to submanifolds of the Coulomb branch where $N-n$ monopoles of the $\mathcal{N}=2$ theory are massless (physically, the constraint is imposed because the massless monopoles get masses and vevs when W is turned on). The perturbation by W lifts all vacua of the $\mathcal{N}=2$ Coulomb moduli space, except for a finite number which survive.

It is convenient to introduce, for each $\mathcal{N}=2$ vacuum, a classical $N\times N$ matrix Φ_{cl} such that $\langle \mathrm{Tr}\ \Phi^r\rangle = \mathrm{Tr}\ \Phi_{cl}^r$ for $r=1,\ldots,N$. Moreover, we set $u_r = \frac{1}{r}\mathrm{Tr}\ \Phi_{cl}^r$ for all positive integers r. For $r=1,\ldots,N$, the u_r are independent and are the usual order parameters of the $\mathcal{N}=2$ theory. For $r>N$, both $\langle \mathrm{Tr}\ \Phi^r\rangle$ and $\mathrm{Tr}\ \Phi_{cl}^r$ can be expressed in terms of the u_r of $r\le N$, but for $r\ge 2N$, as shown in appendix A of [17], they are unequal.

The classical vacua of this theory are obtained by setting all eigenvalues of Φ and Φ_{cl} equal to roots of $W'(z) = \sum_{r=0}^{k} g_r z^r$. In the present section, we will take the degree of the superpotential to be $k+1\le N$, so that the u_r that appear in the superpotential are independent and $\langle \mathrm{Tr}\ \Phi^r\rangle = \mathrm{Tr}\ \Phi_{cl}^r$. In appendix A, we generalize to arbitrary k.

At a generic point on the Coulomb branch of the $\mathcal{N}=2$ theory, the low energy gauge group is $\mathrm{U}(1)^N$. We want to study vacua in which the perturbation by W leaves only $\mathrm{U}(1)^n$ gauge group at low energies, with $n \leq k$. This occurs if the remaining degrees of freedom become massive when $W \neq 0$ because of the condensation of $N-n$ mutually local magnetic monopoles. This can happen only at points where (at $W'=0$) the monopoles are massless. Monopole condensation for $W \neq 0$ can be seen by including the $N-n$ monopole hypermultiplets in the superpotential,

$$W_{\text{eff}} = \sum_{l=1}^{N-n} M_l(u_k) q_l \widetilde{q}_l + \sum_{r=0}^{k} g_r u_{r+1} \,. \tag{2.22}$$

Here q_l, $\widetilde{q}_l$ are the monopole fields and $M_l(u_k)$ is the mass of l^{th} monopole as a function of the u_k. In a supersymmetric vacuum, the variation of W_{eff} with respect to q_l and $\widetilde{q}_l$ is zero. However, for suitable g_r, the variation with respect to the u_k's does not allow $q_l\widetilde{q}_l$ to be zero. Therefore the mass of the monopoles has to vanish, i.e. $M_l(\langle u_k \rangle) = 0$ for $l = 1, \ldots, N-n$.

The masses M_l are known to be equal to periods of a certain meromorphic one-form over some cycles of the $\mathcal{N}=2$ hyperelliptic curve,

$$y^2 = P_N^2(x) - 4\Lambda^{2N} \tag{2.23}$$

where $P_N(x) = \det(x - \Phi_{cl})$ is a polynomial of degree N.

In [4], it was shown that it is more convenient to use the fact that at points with $N-n$ mutually local massless monopoles, the $\mathcal{N}=2$ curve degenerates as follows:

$$y^2 = P_N^2(x) - 4\Lambda^{2N} = F_{2n}(x) H_{N-n}^2(x) \,. \tag{2.24}$$

This factorization is satisfied on an n-dimensional subspace of the Coulomb branch on which the superpotential should be extremized to find the points that preserve $\mathcal{N}=1$ supersymmetry.

The condition (2.24) can easily be incorporated by means of Lagrange multipliers [4]. Using such a superpotential, it was shown in [4] that on shell and when the degree k of $W'(x)$ is equal to n, the highest $n+1$ coefficients of $F_{2n}(x)$ are given in terms of $W'(x)$ as follows,

$$F_{2n}(x) = \frac{1}{g_n^2} W'(x)^2 + \mathcal{O}(x^{n-1}) \,. \tag{2.25}$$

Assuming that W is given and the problem is to determine the $\mathcal{N}=2$ vacuum or equivalently $P_N(x)$, (2.25) is used as follows. (2.25) determines $F_{2n}(x)$ in terms of n unknown coefficients. These are then determined, together with the desired P_N, by asking for existence of a polynomial H_{N-n} such that $P_N^2 - 4\Lambda^{2N} = F_{2n} H_{N-n}^2$, as in (2.24). It is also often convenient to study the inverse problem: starting with an $\mathcal{N}=2$ vacuum with $N-n$ massless monopoles — in other words, a P_N that enables the factorization in (2.24) for suitable H_{N-n} and F_{2n} — one asks what superpotential W would lead to this vacuum. For $k=n$, this problem has a unique solution (modulo an irrelevant additive constant) since if $F_{2n}(x)$ is known, (2.25) determines $W'(x)$.

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The main result of this subsection is to give the generalization of (2.25) to $k > n$, or in other words to the case in which some of the N_i are zero. We repeat the proof of (2.25) given in [4], using a slightly different way to introduce the constraints. In appendix A, we make a further generalization to arbitrary k, dropping the restriction $k \leq N-1$.

Let us add to the superpotential constraints imposing the factorization (2.24). It is useful to write $H_{N-n}(x) = \prod_{i=1}^{N-n}(x - p_i)$; i.e. p_i are the locations of the double roots of $P_N^2 - 4\Lambda^{2N}$. Then we take

$$\begin{aligned} W_{\text{eff}} &= \sum_{r=0}^{k} g_r u_{r+1} + \sum_{i=1}^{N-n} \left(L_i \oint \frac{P_N(x) - 2\epsilon_i \Lambda^N}{x - p_i} dx + B_i \oint \frac{P_N(x) - 2\epsilon_i \Lambda^N}{(x-p_i)^2} dx \right) \\ &= \sum_{r=0}^{k} g_r u_{r+1} + \sum_{i=1}^{N-n} \left(L_i \left(\oint \frac{P_N(x)}{x - p_i} dx - 2\epsilon_i \Lambda^N \right) + B_i \oint \frac{P_N(x)}{(x-p_i)^2} dx \right). \end{aligned} \tag{2.26}$$

Here $P_N(x)$ is a function of u_r. L_i and B_i are Lagrange multipliers imposing the constraints, $\epsilon_i = \pm 1$ and the contour of integration is a large curve around $x = \infty$ that encloses all p_i's. We also have included a factor of $\frac{1}{2\pi i}$ in the definition of the symbol $\oint$ in order to avoid cluttering of equations.

The variation with respect to B_i leads to

$$0 = \oint \frac{P_N(x)}{(x-p_i)^2} dx = P_N'(p_i) = P_N(p_i) \operatorname{Tr} \frac{1}{p_i - \Phi_{cl}} \tag{2.27}$$

(The last step in (2.27) follows from the fact that $P_N(x) = \det(x - \Phi_{cl}) = \prod_i (x - \phi_{cl}^i)$, where ϕ_{cl}^i are the eigenvalues of Φ_{cl}. Since $P_N(p_i) \neq 0$ (given (2.24) and the fact that $H_{N-n}(p_i) = 0$), we learn that

$$\operatorname{Tr} \frac{1}{p_i - \Phi_{cl}} = 0 \tag{2.28}$$

We can use either (2.27) or (2.28) to solve for p_i in terms of the u_r, but let us not do it now. Equation (2.27) or (2.28) will be used in appendix B.

Minimization of $W_{\text{eff}} = W_{\text{eff}}(u_r, p_i, L_i, B_i; g_r, \Lambda)$ leads to $W_{low} = W_{low}(g_r, \Lambda)$. However, we will not attempt to carry this out. Instead, we will try to get information about F_{2n} at the minimum by continuing to study the field equations.

Let us consider variations of (2.26) with respect to p_i,

$$\oint P_N(x) \left(\sum_{i=1}^{N-n} \frac{L_i}{(x-p_i)^2} + \frac{2B_i}{(x-p_i)^3} \right) dx = 2B_i \oint \frac{P_N(x)}{(x-p_i)^3} dx = 0 \tag{2.29}$$

where we used (2.27). Assuming that (2.24) does not have any triple or higher roots, (2.29) implies that $B_i = 0$.

Let us now consider variations with respect to u_r. We use

$$P_N(x) = \exp\left(\operatorname{Tr} \ln(x - \Phi_{cl})\right) = x^N \exp\left(-\sum_{r=0}^{\infty} \frac{u_r}{x^r} \right). \tag{2.30}$$

Since $P_N(x)$ is a polynomial in x, (2.30) can be used to express u_r with $r > N$ in terms of u_r with $r \leq N$ by imposing the vanishing of negative powers of x. (See appendix A.) We can also write (2.30) as

$$P_N(x) = \left[x^N \exp\left(-\sum_{r=0}^{\infty} \frac{u_r}{x^r} \right) \right]_+ \tag{2.31}$$

only in terms of the independent u_r, those with $r \leq N$.[7] The derivative of (2.31) with respect to u_r is

$$\frac{\partial P_N(x)}{\partial u_r} = -\left[\frac{P_N(x)}{x^r} \right]_+ \tag{2.32}$$

We are now ready to differentiate $W_{\rm eff}$ with respect to u_r:

$$g_{r-1} = \oint \left[\frac{P_N(x)}{x^r} \right]_+ \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} dx = \oint \frac{P_N(x)}{x^r} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} dx \tag{2.33}$$

It is convenient to multiply (2.33) with z^{r-1} and to sum over r (z is inside the contour of integration)

$$W'(z) = \oint \frac{P_N(x)}{x-z} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} dx\,. \tag{2.34}$$

We define the polynomial $Q(x)$ in terms of

$$\sum_{i=1}^{N-n} \frac{L_i}{x-p_i} = \frac{Q(x)}{H_{N-n}(x)}\,. \tag{2.35}$$

and write (2.34) as

$$W'(z) = \oint \frac{Q(x)}{H_{N-n}(x)} \frac{P_N(x)}{x-z} dx\,. \tag{2.36}$$

Since $W'(z)$ is a polynomial of degree k, the polynomial $Q(x)$ is of degree $k-n$, and we will denote it as Q_{k-n} (its definition (2.35) determined it as a polynomial of degree $N-n-1 \geq k-n$).

Finally, notice from (2.24) that,

$$P_N(x) = \sqrt{F_{2n}(x)}H_{N-n}(x) + \mathcal{O}(x^{-N}).$$

So we can freely replace $P_N(x)$ by $\sqrt{F_{2n}(x)}H_{N-n}(x)$ in (2.36) to get,

$$W'(z) = \oint \frac{Q_{k-n}(x)\sqrt{F_{2n}(x)}}{x-z} dx\,. \tag{2.37}$$

We can easily recognize (2.37) if it is written

$$W'(z) = \oint \frac{y_m(x)}{x-z} dx \tag{2.38}$$

[7]The symbol $[A]_+$ denotes the polynomial part of a Laurent series A.

with

$$y_m^2(x) = F_{2n}(x) Q_{k-n}^2(x). \tag{2.39}$$

(2.39) is the matrix model curve (for a case in which only n of a possible k cuts have "opened up"), and (2.38) is the corresponding "equation of motion" for the matrix model eigenvalues.

From (2.38) we conclude that (2.25) has to be generalized to

$$F_{2n}(x) Q_{k-n}^2(x) = W_k'(x)^2 + \mathcal{O}\left(x^{k-1}\right). \tag{2.40}$$

In the case when $k = n$, we recover (2.25) with $Q_0 = g_n$. (2.40) can be used just like (2.25) to determine the $\mathcal{N} = 2$ vacuum if W is given. With W known, (2.40) determines F_{2n}, assuming its leading coefficient is normalized to 1, in terms of n parameters, which are then determined, along with P_N and H_{N-n}, by requiring the factorization (2.24). For the inverse problem, things are a little different. If we start with an $\mathcal{N} = 2$ vacuum, so that P_N, F_{2n}, and H_{N-n} are known, and we want to constrain W, then (2.40) actually leaves $k - n$ free parameters in the determination of W. In the classical limit, the meaning of this is that if all eigenvalues of Φ are placed at n zeroes of W', then the expectation values $\langle \mathrm{Tr}\ \Phi^r \rangle$ are entirely independent of the other $n - k$ zeroes.

In appendix A, we prove that the generalization of (2.40) for arbitrarily large k is

$$F_{2n}(x) \widetilde{Q}_{k-n}^2(x) = W_k'(x)^2 + \mathcal{O}\left(x^{k-1}\right) \tag{2.41}$$

where

$$\widetilde{Q}_{k-n}(x) = V_{k-N}(x) H_{N-n}(x) + Q_{N-n-1}(x)$$

and $V_{k-N}(x)$ is a polynomial of degree $k - N$ whose coefficients are new Lagrange multipliers.

Note that (2.40) and (2.41) have the same structure. Moreover, they provide enough equations, as in the case of (2.25), to determine the vacuum if W is given, or to determine W modulo a physically sensible redundancy in the case of the inverse problem.

It is interesting to note that this result is consistent with the fact that the matrix model does not have information about the rank of the field theory gauge group $\mathrm{U}(N)$. It only knows about the number of cuts, i.e. the number of non-zero N_i.

See appendix A for the generalization to superpotentials of arbitrary degree.

2.3 The multiplication map and the confinement index

In [4], a construction was described that, for any positive integer t, maps vacua of the $\mathrm{U}(N)$ theory with a given superpotential W to vacua of the $\mathrm{U}(tN)$ theory with the same superpotential. This operation maps classical limits with unbroken $\prod_{i=1}^n \mathrm{U}(N_i)$ to classical limits with unbroken $\prod_{i=1}^n \mathrm{U}(tN_i)$ and so in particular leaves fixed the rank n of the low energy gauge group. We now review this construction and show that it multiplies the confinement index by t and that all vacua of $\mathrm{U}(tN)$ with confinement index t arise from the Coulomb vacua of $\mathrm{U}(N)$ under this multiplication map. This implies in particular that, for any N, the only "new" vacua, which cannot be deduced from a $\mathrm{U}(N/t)$ vacuum (for some t) via the multiplication map, are the Coulomb vacua.

A key role is played in the analysis by the polynomials [20] that describe those $\mathcal{N}=2$ vacua of the U(t) theory that have the maximum number of massless monopoles, namely $t-1$. These are the vacua that survive when the $\mathcal{N}=2$ theory is perturbed by a quadratic superpotential $W=\frac{1}{2}\,\mathrm{Tr}\,\Phi^2$; this perturbation removes Φ from the low energy theory, whereupon the SU(t) gauge theory becomes massive at low energies, with t vacua. From the $\mathcal{N}=2$ point of view, this mass generation in the SU(t) sector arises from $t-1$ condensed monopoles. To have $t-1$ massless monopoles before the superpotential is turned on, the right hand side of the equation of the $\mathcal{N}=2$ curve $y^2=P_t(x)^2-4\Lambda^{2t}$ must admit a special factorization,

$$P_t^2(x)-4\Lambda^{2t}=F_2(x)H_{t-1}^2(x)\,, \tag{2.42}$$

where we can assume that P_t, F_2, and H_{t-1} all have leading coefficient 1. Moreover, setting $F_2=(W')^2+f_0$, we see that if $W=x^2/2$, we want $F_2=x^2-a\Lambda^2$ for some constant a. This problem leads to t possibilities for P_t and H_{t-1}, found by using Chebyshev polynomials.

Chebyshev polynomials of the first and second kind of degree t and $t-1$ respectively are defined as follows. (See appendix C for some explicit formulas.) Set $x=\cos\theta$ and let

$$\mathcal{T}_t(x)=\cos(t\theta) \qquad \mathcal{U}_{t-1}(x)=\frac{1}{t}\frac{d\mathcal{T}_t}{dx}(x)=\frac{\sin(t\theta)}{\sin\theta}\,, \tag{2.43}$$

the point being that $\cos(t\theta)$ and $\sin(t\theta)/\sin\theta$ are both polynomials in x. Some identities that can easily be checked are

$$\begin{aligned}\mathcal{T}_t(x)^2-1 &= (x^2-1)\mathcal{U}_{t-1}(x)^2\\ \mathcal{T}_t(x) &= \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^t+\left(x-\sqrt{x^2-1}\right)^t\right).\end{aligned} \tag{2.44}$$

From the second identity[8] it is clear that the coefficient of x^t in $T_t(x)$ is 2^{t-1}.

The solutions to (2.42) are[9]

$$P_t(x)=2\Lambda^t\eta^t\mathcal{T}_t\left(\frac{x}{2\eta\Lambda}\right),\quad F_2(x)=x^2-4\eta^2\Lambda^2\,,\quad H_{t-1}(x)=\eta^{t-1}\Lambda^{t-1}\mathcal{U}_{t-1}\left(\frac{x}{2\eta\Lambda}\right), \tag{2.45}$$

with η a $2t$-th root of unity, i.e. $\eta^{2t}=1$. Using the fact that $P_t(-x)=(-1)^tP_t(x)$, one can see that (2.45) is a function only of η^2 and hence gives us precisely the expected t solutions of the factorization problem.

Using this, we can go back to the original problem of mapping U(N) solutions with a given superpotential, and some renormalization scale Λ_0, to U(tN) solutions with the same superpotential and renormalization scale Λ.

By assumption, we have polynomials $P_N(x)$, $F_{2n}(x)$ and $H_{N-n}(x)$, all with leading coefficient 1, obeying the following relations:

$$\begin{aligned}P_N^2(x)-4\Lambda_0^{2N} &= F_{2n}(x)H_{N-n}^2(x)\\ F_{2n}(x) &= (W')^2+f_{n-1}(x)\,.\end{aligned} \tag{2.46}$$

[8] To prove this, write $\cos(t\theta)=\frac{1}{2}(e^{t\theta i}+e^{-t\theta i})$

[9] We are making a slight refinement of the formulas as they have been presented previously.

From (2.44), one can show that $2\Lambda^{tN}\eta^t\mathcal{T}_t(\widetilde{x})$ with $\widetilde{x} = \frac{1}{2\eta\Lambda^N}P_N(x)$ satisfies

$$\begin{aligned} 2\Lambda^{tN}\eta^t\mathcal{T}_t(\widetilde{x}) &= x^{tN} + \mathcal{O}\left(x^{tN-1}\right) \\ \left(2\Lambda^{tN}\eta^t\mathcal{T}_t(\widetilde{x})\right)^2 - 4\Lambda^{2tN} &= \left(P_N^2(x) - 4\eta^2\Lambda^{2N}\right)\left(\eta^{t-1}\Lambda^{N(t-1)}\mathcal{U}_{t-1}(\widetilde{x})\right)^2 . \end{aligned} \tag{2.47}$$

Using (2.46) in the last equation with the identification $\Lambda_0^{2N} = \eta^2\Lambda^{2N}$, one finds

$$\left(2\Lambda^{tN}\eta^t\mathcal{T}_t(\widetilde{x})\right)^2 - 4\Lambda^{2tN} = F_{2n}(x)H_{N-n}^2(x)\left(\eta^{t-1}\Lambda^{N(t-1)}\mathcal{U}_{t-1}(\widetilde{x})\right)^2 . \tag{2.48}$$

The factorization problem for vacua of the U(tN) problem with low energy gauge group U$(1)^n$ is

$$P_{tN}^2(x) - 4\Lambda^{2tN} = \widetilde{F}_{2n}(x)H_{tN-n}^2(x) . \tag{2.49}$$

Comparison of (2.48) and (2.49) leads to the identification of

$$\begin{aligned} P_{tN}(x) &= 2\Lambda^{tN}\eta^t\mathcal{T}_t\left(\frac{P_N(x)}{2\eta\Lambda^N}\right) \\ \widetilde{F}_{2n}(x) &= F_{2n}(x) \\ H_{tN-n}(x) &= \eta^{t-1}\Lambda^{N(t-1)}H_{N-n}(x)\mathcal{U}_{t-1}\left(\frac{P_N(x)}{2\eta\Lambda^N}\right) \end{aligned} \tag{2.50}$$

as solutions of (2.49). Since $\widetilde{F}_{2n} = F_{2n}$, the vacua constructed this way for the U(tN) theory have the same superpotential as the vacua of the U(N) theory.

For a given superpotential W, the U(N) theory has a finite number of vacua with given n. This number is independent of Λ_0. For t different values of the U(N) parameter Λ_0^{2N}, namely

$$\Lambda_0^{2N} = \eta^2\Lambda^{2N} , \tag{2.51}$$

we have been able to construct U(tN) vacua. Thus, this construction gives t times as many U(tN) vacua as there are U(N) vacua. The relation (2.51) shows that, to obtain U(tN) vacua in which the θ-angle equals θ_{tN}, the analogous angle θ_N of the U(N) theory must be $\theta_N = \theta_{tN} + 2\pi l/t$ for some integer l.

This construction gives t branches in U(tN) for every branch in U(N), since the construction depends on the choice of η^2. Since we will show that the construction multiplies the confinement index by t, the counting of branches is in accord with the observation in section 2.1 that Wilson and 't Hooft loops can distinguish t types of branch with confinement index t.

Unbroken gauge group. As a check of our formalism, let us see how the tN vacua with unbroken U(tN) arise by applying the multiplication by t map to the unbroken vacua of U(N).

In this case, $P_N(x)$ satisfies (2.46) with $n = 1$, i.e. it is given by,

$$P_N(x) = 2\Lambda_0^N\epsilon^N\mathcal{T}_N\left(\frac{x}{2\epsilon\Lambda_0}\right)$$

with $\epsilon^{2N} = 1$.

Using this in (2.50), we get

$$P_{tN}(x) = 2\Lambda^{tN}\eta^t\mathcal{T}_t\left(\frac{\Lambda_0^N\epsilon^N}{\eta\Lambda^N}\mathcal{T}_N\left(\frac{x}{2\epsilon\Lambda_0}\right)\right) \tag{2.52}$$

Using $\Lambda_0^{2N} = \eta^2\Lambda^{2N}$, one can easily check that the combination $\frac{\Lambda_0^N\epsilon^N}{\eta\Lambda^N}$ squares to one and can be pulled out of $\mathcal{T}_t$ giving a factor of $\left(\frac{\Lambda_0^N\epsilon^N}{\eta\Lambda^N}\right)^t$. This implies that (2.52) can be written as

$$P_{tN}(x) = 2(\epsilon\Lambda_0)^{tN}\mathcal{T}_t\left(\mathcal{T}_N\left(\frac{x}{2\epsilon\Lambda_0}\right)\right) \tag{2.53}$$

Let us introduce $\widetilde{\sigma}$ obeying $\widetilde{\sigma}^{2N} = \eta^2$ as well as $\Lambda_0 = \widetilde{\sigma}\Lambda$, and define $\sigma = \epsilon\widetilde{\sigma}$. Finally, using the identity $\mathcal{T}_{tN}(x) = \mathcal{T}_t(\mathcal{T}_N(x))$ in (2.53), one gets

$$P_{tN}(x) = 2\Lambda^{tN}\sigma^{tN}\mathcal{T}_{tN}\left(\frac{x}{2\sigma\Lambda}\right).$$

Recalling that η is a $2t$-root of unity we get that $\widetilde{\sigma}$ is a $2tN$-root of unity, and so is σ since ϵ is a $2N$-root of unity. This, as we wished to show, is the expected solution of the U(tN) problem.

Generating function of observables Tr Φ^k**.** In section 2.1, a criterion for confinement was given in terms of the operator-valued one-form constructed from the generating function of Tr Φ^k,

$$T(x) = \left\langle \mathrm{Tr}\, \frac{dx}{x-\Phi}\right\rangle. \tag{2.54}$$

Let us denote by $T_{tN}(x)$ and $T_N(x)$ the one forms of U(tN) and U(N) respectively.

Recall that (according to appendix A of [17])

$$\left\langle \mathrm{Tr}\, \frac{1}{x-\Phi}\right\rangle = \frac{P'_{tN}(x)}{\sqrt{P^2_{tN}(x) - 4\Lambda^{2tN}}}. \tag{2.55}$$

From (2.43) and (2.50), we get

$$P'_{tN}(x) = 2\eta^t\Lambda^{tN}\frac{d\widetilde{x}}{dx}\mathcal{T}'_t(\widetilde{x}) = t\eta^{t-1}P'_N(x)\Lambda^{N(t-1)}\mathcal{U}_{t-1}(\widetilde{x}). \tag{2.56}$$

Using this, (2.47), and (2.50), we get

$$\left\langle \mathrm{Tr}\, \frac{1}{x-\Phi}\right\rangle = t\frac{P'_N(x)}{\sqrt{P^2_N(x) - 4\Lambda_0^{2N}}}. \tag{2.57}$$

If we denote by Φ_0 the adjoint field in the U(N) theory, then (2.57) can be written as

$$\left\langle \mathrm{Tr}\, \frac{1}{x-\Phi}\right\rangle = t\left\langle \mathrm{Tr}\, \frac{1}{x-\Phi_0}\right\rangle \tag{2.58}$$

or equivalently $T_{tN}(x) = tT_N(x)$.

So the multiplication by t map multiplies all periods of T by t. Since the confinement index is the greatest common divisor of the periods, it follows that this map multiplies the confinement index by t. Since the N_i are certain periods of T, it follows also that the map multiplies the N_i by t, as originally shown in [4].

All the confining vacua. As a special case, if we start with a Coulomb vacuum in $\mathrm{U}(N)$, the multiplication by t map produces vacua with confinement index t in $\mathrm{U}(tN)$.

The converse is also true: all vacua in $\mathrm{U}(tN)$ with confinement index t arise in this way from Coulomb vacua in $\mathrm{U}(N)$. This can be shown by counting vacua. Because of the freedom to choose η, the multiplication by t map produces t times as many vacua with confinement index t in $\mathrm{U}(tN)$ as there are Coulomb vacua in $\mathrm{U}(N)$. Let us show that this is the correct number.

As explained in section 2.1, a Coulomb vacuum in $\mathrm{U}(N)$ with a classical limit based on $\prod_i \mathrm{U}(N_i)$ is specified by picking some integers r_i with $0 \le r_i \le N_i - 1$ and such that the collection of the N_i and $b_i = r_i - r_{i+1}$ are relatively prime. Instead, to give a vacuum in $\mathrm{U}(tN)$ with classical limit based on $\prod_i \mathrm{U}(tN_i)$ and with confinement index t, one must pick some integers $\widetilde{r}_i$, $0 \le \widetilde{r}_i \le tN_i - 1$, such that the collection tN_i and $\widetilde{b}_i = \widetilde{r}_i - \widetilde{r}_{i+1}$ have greatest common divisor t. Any such $\widetilde{r}_i$ are uniquely of the form $\widetilde{r}_i = u + tr_i$ with $0 \le u \le t-1$ (and that formula works for any u and for any r_i that solve the problem in $\mathrm{U}(N)$). As there are t choices of u, the number of $\mathrm{U}(tN)$ vacua with confinement index t should indeed be t times the corresponding number of $\mathrm{U}(N)$ Coulomb vacua.

2.4 Further strong coupling generalities

Here we will consider some additional issues from the strong coupling point of view, in which one starts with the solution of the $\mathcal{N}=2$ theory and treats the superpotential as a small perturbation. The $\mathrm{SU}(N)$ dynamics is described by the curve Σ:

$$y^2 = P_N(x)^2 - 4\Lambda^{2N} \,. \tag{2.59}$$

Here $P_N(x) = \det(x - \Phi_{cl})$. Generically, the expectation value of Φ breaks $\mathrm{SU}(N)$ to $\mathrm{U}(1)^{N-1}$, and Σ has genus $N-1$. Its jacobian determines the effective couplings of the $N-1$ low energy abelian vector multiplets. Including the $\mathrm{U}(1)$ center of $\mathrm{U}(N)$, which is free and not described by Σ, the low energy unbroken gauge group is $\mathrm{U}(1)^N$.[10]

If the right hand side of Σ has w double roots, this corresponds to existence of w independent massless monopoles. After small perturbation by a generic superpotential, reducing the supersymmetry to $\mathcal{N}=1$, the monopoles will condense, reducing the low energy gauge group from $\mathrm{U}(1)^N$ to $\mathrm{U}(1)^{N-w}$. So to describe a vacuum of the $\mathcal{N}=1$ theory with low energy gauge group $\mathrm{U}(1)^n$, we need $N-n$ double zeroes of the polynomial $P_N(x)^2 - 4\Lambda^{2N}$.

This polynomial factorizes

$$P_N^2 - 4\Lambda^{2N} = \left(P_N + 2\Lambda^N\right)\left(P_N - 2\Lambda^N\right). \tag{2.60}$$

The two factors $P_N + 2\Lambda^N$ and $P_N - 2\Lambda^N$ never vanish simultaneously, so a double root is a double root of one factor or the other. To get $w = N-n$ double roots, we therefore place s_+ double roots in one factor and s_- in the other factor, with $w = s_+ + s_-$. If w

[10] In this paper, we will not require Φ to be traceless. So Φ describes all chiral superfields, including the $\mathcal{N}=2$ partner of the center of $\mathrm{U}(N)$. But the jacobian of Σ describes the gauge couplings only for the $\mathrm{SU}(N)$ gauge fields. When a superpotential is turned on, the trace part of Φ interacts with the other fields, but the $\mathrm{U}(1)$ gauge field remains free.

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is given, there are $w+1$ ways to pick s_+ and s_- with $w = s_+ + s_-$, so if all possibilities actually occur, there are (at least) $w+1$ branches of the moduli space of vacua on which the rank of the unbroken group is $n = N - w$. In the examples that we have looked at, all values of s_+ and s_- with $2s_+, 2s_- \leq N$ are possible — they all are realized for some N^{th} order polynomial P_N. (Note that every $P_N(x)$ is of the form $P_N(x) = \det(x - \Phi_{cl})$ for some Φ_{cl}.) We actually will meet cases with several branches for the same s_+ and s_-.

This multiplicity of branches cannot, as far as we know, be explained using conventional order parameters. The exchange $s_+ \leftrightarrow s_-$ can be understood as a discrete chiral symmetry of the $\mathcal{N}=2$ theory under which Λ^N changes sign. In some cases (with w close to N) that we will study in section 3, confinement goes a long way in explaining why there are several branches. However, in general, chiral symmetries and confinement (and any conventional order parameters we know) do not suffice to explain why there are so many branches. In section 4, we study examples in which w is small compared to N and show that the branches with different s_+ and s_- can be characterized according to which holomorphic functions of the chiral order parameters vanish on a given branch.

Now let us explain something that will be useful background for our computations in section 3. Consider a family of $\mathcal{N}=2$ vacua with $N-n$ massless monopoles and given s_+, s_- (and $s_+ + s_- = N - n$). As we take $\Lambda \to 0$, can the theory converge to a classical limit in which the unbroken gauge group is $\prod_{i=1}^n \mathrm{U}(N_i)$?

If such a limit does exist, then for very small Λ, the $\mathrm{U}(N_i)$ theories approximately decouple from each other. Each has its own polynomial P_{N_i} and its own mass scale Λ_i, and its own curve $y_i^2 = P_{N_i}(x)^2 - 4\Lambda_i^{2N_i}$. If each $\mathrm{U}(N_i)$ theory has w_i massless monopoles, we need $\sum_i w_i = N-n$ so that the full theory has $N-n$ massless monopoles. But in $\mathrm{U}(N_i)$, the maximum possible number of massless monopoles is $N_i - 1$, and since $\sum_{i=1}^n (N_i - 1) = N - n$, we must in fact have $w_i = N_i - 1$ for all i.

The particular polynomials $P_{N_i}(x)$ such that $P_{N_i}^2 - 4\Lambda_i^{2N_i}$ has $N_i - 1$ double roots are known explicitly in terms of Chebyshev polynomials [20] and were used in section 2.3. Even without recalling the details, we can determine what numbers s_{+i} and s_{-i} of double roots of $P_{N_i} \pm 2\Lambda_i^{2N_i}$ are possible if $s_{+i} + s_{-i} = N_i - 1$. A polynomial of degree N_i has at most $N_i/2$ double roots, so $s_{+i}, s_{-i} \leq N_i/2$. If N_i is odd, the only possibility is $s_{+i} = s_{-i} = (N_i - 1)/2$, and if N_i is even, there are two cases $(s_{+i}, s_{-i}) = (N_i/2, N_i/2 - 1)$ or $(N_i/2 - 1, N_i/2)$.

Finally we can answer our question of which classical limits can exist for given s_+, s_-. A branch of given s_+ and s_- can have a classical limit with unbroken $\prod_{i=1}^n \mathrm{U}(N_i)$ only if one can write $s_+ = \sum_i s_{+i}$, $s_- = \sum_i s_{-i}$, with $0 \leq s_{+i}, s_{-i} \leq N_i/2$ and $s_{+i} + s_{-i} = N_i - 1$. The above argument shows that this condition is necessary; in examples that we have studied, it is also sufficient. For example, in section 3, we will examine in detail the case of $N = 6$ with $n = 2$, so $s_+ + s_- = 4$. For $(s_+, s_-) = (3, 1)$ or $(1, 3)$, our conditions only allow the classical limit $\mathrm{U}(4) \times \mathrm{U}(2)$, in agreement with what we will find. But for $(s_+, s_-) = (2, 2)$, they allow $\mathrm{U}(5) \times \mathrm{U}(1)$, $\mathrm{U}(4) \times \mathrm{U}(2)$, and $\mathrm{U}(3) \times \mathrm{U}(3)$, all of which will duly appear.

Intersections of branches. Now let us consider whether different branches might meet at singularities. For example, can a branch $\mathcal{M}_1$ with $(s_+, s_-) = (a, b)$ meet a branch $\mathcal{M}_2$ with $(s_+, s_-) = (a+1, b-1)$? This will naturally occur at a point with $w+1$ massless monopoles and $(s_+, s_-) = (a+1, b)$. Such a point is on both $\mathcal{M}_1$ and $\mathcal{M}_2$, as well as being on a branch $\mathcal{M}$ with $w+1$ massless monopoles. So in general, distinct branches with "adjacent" values of s_+, s_- will meet at singular points with an extra massless monopole.

Such a singularity corresponds to shrinking a cycle in the Riemann surface of the $\mathcal{N} = 2$ theory. From (2.20) it is clear that the differential T cannot have a pole. Therefore, a cycle can shrink only if the period of T around it vanishes. This leads to restrictions on the possible singularities in a branch. Similar restrictions have been pointed out in [8]. They do not affect the examples we study below because even if $\oint T$ around a certain cycle is not zero, we can find another cycle with vanishing period which can be shrunk.

Here we have described the situation in an $\mathcal{N} = 2$ language. From an $\mathcal{N} = 1$ point of view, the situation is different. For a given branch with given w, the superpotential that leads to a given $\mathcal{N} = 2$ curve on the branch can be found by the recipe of [4], reviewed in section 2.2. (It generically triggers condensation of all the massless monopoles.) For simplicity, we assume that on the branches $\mathcal{M}_1$ and $\mathcal{M}_2$, $k = n$ so the superpotential is uniquely determined. The superpotential needed to get a given $\mathcal{N} = 2$ curve on a branch with $(s_+, s_-) = (a, b)$ is generically different from the superpotential needed to get the same curve on a branch with $(s_+, s_-) = (a+1, b-1)$. So this type of intersection of branches is not really relevant for $\mathcal{N} = 1$, which is our main interest in the present paper.

What is more relevant for $\mathcal{N} = 1$ is that a branch $\mathcal{M}_1$ with $(s_+, s_-) = (a, b)$ meets a branch $\widetilde{\mathcal{M}}$ with $(s_+, s_-) = (a+1, b)$ at a point with an extra double root. (An example was discussed in [8].)[11]

The notion of $\widetilde{\mathcal{M}}$ and $\mathcal{M}_1$ "meeting" may seem confusing since from an $\mathcal{N} = 2$ point of view, $\widetilde{\mathcal{M}}$ is just a subspace of $\mathcal{M}_1$ singled out by existence of an extra double root. So let us elaborate. $\widetilde{\mathcal{M}}$ actually has a reduced rank of the low energy gauge group $n = k-1$, so on this branch the superpotential is not uniquely determined by the $\mathcal{N} = 2$ curve, but depends on one extra complex parameter. This parameter controls the condensation of the massless monopole that appeared when $P_N + 2\Lambda^{2N}$ developed an extra double root. This parameter should be included in describing the $\widetilde{\mathcal{M}}$ branch from an $\mathcal{N} = 1$ point of view. Thus, from an $\mathcal{N} = 1$ point of view, $\widetilde{\mathcal{M}}$ is not a subspace of $\mathcal{M}_1$, and in fact they have the same dimension.

Starting on $\mathcal{M}_1$, the intersection with $\widetilde{\mathcal{M}}$ is obtained by adjusting the $\mathcal{N} = 2$ curve to get an extra double root. Starting from $\widetilde{\mathcal{M}}$ (at a generic point where the extra monopole is condensed), the intersection is reached by varying the superpotential until it becomes the $\mathcal{M}_1$ superpotential for the same $\mathcal{N} = 2$ curve.

For this to make sense, we need to know that the $\mathcal{M}_1$ superpotential of any $\mathcal{N} = 2$ curve with $(s_+, s_-) = (a+1, b)$ is always one of the allowed superpotentials that can lead to that curve on $\widetilde{\mathcal{M}}$. This can readily be shown from the generalized [4] recipe for the superpotential that was presented in section 2.2 (see (2.40) and the discussion following it).

[11] The branch $\widetilde{\mathcal{M}}$ likewise meets the branch $\mathcal{M}_2$ with $(s_+, s_-) = (a, b-1)$; as we have just explained, this occurs with the same $\mathcal{N} = 2$ curve but a different superpotential from the one that leads to $\mathcal{M}_1$ intersecting $\widetilde{\mathcal{M}}$.

Sometimes it will happen that the branch $\widetilde{\mathcal{M}}$ is confining while $\mathcal{M}_1$ is not confining (or has a smaller confinement index). If so, then as one approaches the intersection of the two branches starting on $\widetilde{\mathcal{M}}$, the coefficient of the area law must vanish (as noted in an example in [8]).

Actually, a point with $(s_+, s_-) = (a+1, b)$ is on an $(a+1)$-fold self-intersection of $\mathcal{M}_1$, since there are $a+1$ different ways to forget a double root of $P_N + 2\Lambda^N$, leaving a of them. Such a point is likewise on a b-fold self-intersection of $\mathcal{M}_2$. However, the different branches do not really meet as $\mathcal{N} = 1$ theories, since they would require different superpotentials.

Though in the present paper we focus on double roots of $P_N^2 - 4\Lambda^{2N}$, it is also possible to consider the case of triple or higher order roots. The first example was studied by Argyres and Douglas [21]. At the $\mathcal{N} = 2$ level, before turning on a superpotential, these configurations are believed to correspond to non-trivial critical points. What happens if a superpotential is turned on, breaking to $\mathcal{N} = 1$? A generic superpotential will not be extremal at a point with a triple root, but by tuning parameters to get a superpotential that is extremal at such a point, one can presumably get a nontrivial critical point with $\mathcal{N} = 1$ supersymmetry.

Off-shell interpolation. Most of this paper is devoted to the question of branches of physical vacua. As the parameters in $W(\Phi)$ are varied, the vacua change and we explore the various branches and the smooth interpolation from one vacuum to another within a branch. These are on-shell interpolations. It is interesting to ask whether there exists a lagrangian which allows us to extrapolate between the different branches.

This question can be answered at various levels. The most complete description of the theory is in terms of the microscopic gauge theory. It describes all the vacua of the theory.

A more macroscopic description is the effective lagrangian of [4] for the fields S_i. The vacua are found by solving the equations of motion of S_i. As we vary the parameters in $W(\Phi)$, these vacua change and we find the on-shell interpolation. One of our results is that we can continuously interpolate between vacua with one value of N_i to vacua with other values of N_i. Therefore, the superpotential of [4] has many stationary points corresponding to the different vacua with different N_i on the same branch. We will see examples of this in the next section.

Another type of an effective lagrangian we can consider is the following. The effective lagrangian for a given branch has the form (2.13)

$$\sum_i N_i S_i \left(\log \left(\frac{\Lambda^3}{S_i} \right) + 1 \right) + 2\pi i b_i S_i + \sum_i N_i \mathcal{O}(S_j S_k) \tag{2.61}$$

(we absorbed τ_0 into Λ and $b_1 = 0$). We can replace it with

$$\sum_i S_i \left(\log \left(\frac{\Lambda^{3N_i}}{S_i^{N_i}} \right) + N_i \right) + \sum_i N_i \mathcal{O}(S_j S_k)\,. \tag{2.62}$$

As we explained in section 2.1, this superpotential has more stationary points than (2.61). For small S_i it has $\prod_i N_i$ vacua, rather than a single vacuum. Therefore, it describes different branches, some of them can be confining and others can be in a Coulomb phase.

A superpotential like (2.62) might be a good effective lagrangian for all vacua for a given N_i for small S_i, i.e. at weak coupling. However, it is unlikely to describe the proper interpolation to large S_i where vacua with different values of N_i are present.

3. Examples

In this section, we will analyze several explicit examples demonstrating the phenomena we have discussed. In the first five subsections, we will study examples with gauge group $\mathrm{U}(N)$ for $N = 2, \ldots, 6$ and with low rank of the low energy gauge group. The case that the rank is 1 is too trivial for our purposes; this means that $\mathrm{U}(N)$ is unbroken classically. The corresponding $\mathcal{N} = 2$ curves are described by Chebyshev polynomials and were reviewed in section 2.3. We focus therefore on the next case that the low energy group has rank $n = 2$; thus, classically $\mathrm{U}(N)$ is broken to $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$, and the question arises of whether smooth interpolation can occur between different pairs (N_1, N_2). In section 3.6, we present some special examples with $n > 2$.

For each N, the "new" vacua are the Coulomb vacua, since according to the discussion in section 2.3, the confining vacua with confinement index t are determined by what happened for $\mathrm{U}(N/t)$. In our systematic construction of all vacua with $n = 2$ for given N, we will come across all vacua, both Coulomb and confining. For the range of N we consider, interpolation between different pairs (N_1, N_2) occurs only for Coulomb vacua, simply because (if $t > 1$) N/t is always too small for such interpolation to be possible in $\mathrm{U}(N/t)$.

We will analyze the theory along the following steps. We start by considering the $\mathcal{N} = 2$ $\mathrm{U}(N)$ gauge theory whose Coulomb branch is described by the hyper-elliptic curve

$$y^2 = P_N(x)^2 - 4\Lambda^{2N} = \left(P_N(x) + 2\Lambda^N\right)\left(P_N(x) - 2\Lambda^N\right) \tag{3.1}$$

where the N^{th} order polynomial $P_N(x) = \det(x - \Phi_{cl})$ parameterizes the point in the moduli space. This point is labeled by the N eigenvalues of the matrix Φ_{cl} modulo permutations.

To get $n = 2$, the $\mathcal{N} = 2$ theory must have $N - 2$ massless magnetic monopoles, so the polynomial (3.1) must have $N - 2$ double roots. Since P_N depends on N complex parameters, the subspace on which $P_N^2 - 4\Lambda^{2N}$ has $N-2$ double roots is two-dimensional. As discussed in section 2.4, the distribution of double roots between the two factors in (3.1) is labeled by two integers. There are s_+ double roots in the first factor and s_- double roots in the second factor. The two integers s_+ and s_- must satisfy $s_+ + s_- = N - 2$ and $s_+, s_- \leq N/2$. Different values of s_+ and s_- correspond to different branches.

After picking values of s_+ and s_- we solve the factorization problem

$$\begin{aligned} P_N(x) + 2\Lambda^N &= H_{s_+}^2(x) R_{N-2s_+}(x) \\ P_N(x) - 2\Lambda^N &= \widetilde{H}_{s_-}^2(x) \widetilde{R}_{N-2s_-}(x) \end{aligned} \tag{3.2}$$

with polynomials H_{s_+}, $\widetilde{H}_{s_-}$, R_{N-2s_+} and $\widetilde{R}_{N-2s_-}$, where the subscript of the polynomial denotes its degree. We normalize the coefficient of the highest power of x in each polynomial to be one, and solve for the other coefficients such that

$$H_{s_+}^2(x) R_{N-2s_+}(x) - 4\Lambda^N = \widetilde{H}_{s_-}^2(x) \widetilde{R}_{N-2s_-}(x)\,. \tag{3.3}$$

We will see in the examples below that often there are several distinct branches for the same (s_+, s_-).

The moduli space $\mathcal{M}$ is parametrized by the coefficients in the polynomials H_{s_+}, $\widetilde{H}_{s_-}$, R_{N-2s_+} and $\widetilde{R}_{N-2s_-}$. As we mentioned above, $\mathcal{M}$ is two dimensional. In solving the factorization problem (3.2), we will sometimes shift x by a constant in order to simplify the equations. This constant can trivially be reinstated at the end of the calculation. Ignoring this constant, our moduli space is one dimensional.

We then examine the semiclassical limit $\Lambda \to 0$. Since we limit ourselves to the case $n = 2$, in the classical limit the microscopic $\mathrm{U}(N)$ gauge group is broken to $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$. We want to determine for each branch $\mathcal{M}_\alpha$ of the moduli space what values of N_1 and N_2 are possible.

We will also study singular points on $\mathcal{M}_\alpha$. Triple roots of y^2 lead to Argyres-Douglas points [21]. We will not determine them here. We will focus on singularities in which y^2 has a new double root. These occur when either $R_{N-2s}(x)$ or $\widetilde{R}_{N-2t}(x)$ acquire a double root. This means that these points have $(s_+ + 1, s_-)$ or $(s_+, s_- + 1)$ double roots in the two branches. At these subspaces, there are $N - 1$ massless monopoles. As explained in section 2.4, at these points $\mathcal{M}_\alpha$ meets a branch with $n = 1$, i.e., the fully confined branch that classically has unbroken $\mathrm{U}(N)$.

We then break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by turning on a tree level superpotential which leads to a vacuum at a point on $\mathcal{M}_\alpha$; this lifts the degeneracy of $\mathcal{M}_\alpha$, and only a finite set of vacua survive for given superpotential. We want to determine what superpotential can lead to a given point on the moduli space. Since $n = 2$, the superpotential must have at least two critical points, and to keep things simple we will consider the case of a cubic superpotential, which has exactly two critical points. In section 2.2, we reviewed the recipe of [4] for determining what cubic superpotential, normalized to have leading term $x^3/3$, leads to a given point on the $\mathcal{N} = 2$ moduli space. We take the matrix model curve, which in this case is

$$y_m^2 = R_{N-2s_+}(x)\widetilde{R}_{N-2s_-}(x)\,, \tag{3.4}$$

and we write

$$R_{N-2s_+}(x)\widetilde{R}_{N-2s_-}(x) = W'(x)^2 + f_1(x)\,. \tag{3.5}$$

Recall that $s_+ + s_- = N - 2$, and therefore the left hand side of (3.5) is a quartic polynomial. (3.5) determines W', and hence determines W up to an irrelevant additive constant. Moreover, the value of the gluino bilinear S can be read off from $f_1(x)$; if $f_1(x) = Ax + B$, then $S = S_1 + S_2 = -A/4$.

The intersection points of different branches are points where y_m^2 has a double root and therefore equation (3.4) describes a genus zero curve. This is consistent with the fact that such points are also on the $n = 1$ branch because, as explained in section 2.2, when $n < k$ (here $n = 1 < k = 2$), the matrix model curve has double zeros. The physics of these points in the $\mathcal{N} = 1$ theory is interesting. The low energy spectrum at these points consists of the two $\mathrm{U}(1)$ multiplets which exist at generic points in $\mathcal{M}_\alpha$ as well as another massless monopole. Moving away from this point along $\mathcal{M}_\alpha$, this monopole acquires a mass. Moving away from this point through the condensation of this monopole takes us to the fully confining vacua with unbroken $\mathrm{U}(N)$ and only a single massless $\mathrm{U}(1)$ multiplet.

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If we denote the critical points of W as x_1 and x_2, we have

$$W' = (x - x_1)(x - x_2)\,. \tag{3.6}$$

The physics is essentially unchanged by adding a constant to x, which shifts x_1 and x_2 by a constant. The invariant information contained in the choice of W', modulo shifting x and exchanging the two critical points, is contained in the parameter

$$\Delta = (x_1 - x_2)^2\,. \tag{3.7}$$

It is possible and sometimes convenient to shift x so that a superpotential of given Δ takes the form

$$W = \frac{x^3}{3} - \frac{\Delta}{4}x\,. \tag{3.8}$$

W is odd under the transformation $x \to -x$, which thus corresponds to an R-symmetry. (Of course, the symmetry is present whether or not we shift x to put W in the form (3.8).) The critical points x_1, x_2 are exchanged by the symmetry, so a choice of classical vacuum with $N_1 \neq N_2$ spontaneously breaks the symmetry. In some examples, the symmetry exchanges different branches $\mathcal{M}_\alpha$, and in other cases, it acts within a fixed branch.

The matrix model formulas can be used to compute $T(x)$. Though we will not do this in detail, it could be done as follows. Following [17],

$$T(x) = \left\langle \mathrm{Tr}\,\frac{dx}{x - \Phi}\right\rangle = \frac{P_N'(x)}{\sqrt{P_N^2(x) - 4\Lambda^{2N}}}dx\,. \tag{3.9}$$

From (3.2)

$$\begin{aligned} P_N'(x) &= H_{s_+}(x)\left(2H_{s_+}'(x)R_{N-2s_+}(x) + H_{s_+}(x)R_{N-2s_+}'(x)\right)\\ &= \widetilde{H}_{s_-}(x)\left(2\widetilde{H}_{s_-}'(x)\widetilde{R}_{N-2s_-}(x) + \widetilde{H}_{s_-}(x)\widetilde{R}_{N-2s_-}'(x)\right)\\ &= -\frac{1}{4}c(x)H_{s_+}(x)\widetilde{H}_{s_-}(x)\,. \end{aligned} \tag{3.10}$$

Since we learn from the first two expressions that $P_N'(x)$ is divisible by $H_{s_+}(x)$ and $\widetilde{H}_{s_-}(x)$, $c(x)$ must be a polynomial (the factor of $-1/4$ is in order to agree with the conventions of [17]). Therefore (3.9) can be written as

$$T(x) = -\frac{c(x)}{4y_m}dx \tag{3.11}$$

which leads to the identification $c(x) = -4\left[W'(x)T(x)\right]_+ = -4\,\mathrm{Tr}\,\frac{W'(x) - W'(\Phi)}{x - \Phi}$ in [17].

In some of our examples, we will find $S = 0$, and find that this can be interpreted as a consequence of the symmetry. However, the symmetry is special to a cubic superpotential. A generic superpotential of higher degree does not have such a symmetry and it can still lead on certain branches to $S = 0$, as we will demonstrate in section 3.6. Lacking such a symmetry, vanishing of S on a branch of vacua would not be natural in a nonsupersymmetric field theory. With supersymmetry, such "unnatural" phenomena can be dictated by holomorphy. In section 4, we discuss more elaborate examples of such phenomena.

Let us consider the examples.

3.1 U(2)

Our first example is a U(2) gauge theory. The hyper-elliptic curve (3.1) is

$$y^2 = \left(P_2(x) + 2\Lambda^2\right)\left(P_2(x) - 2\Lambda^2\right). \tag{3.12}$$

The factorization problem (3.2) in this case is quite trivial, since to get $n = 2$ we do not need double roots at all — just the classical symmetry breaking $\mathrm{U}(2) \to \mathrm{U}(1) \times \mathrm{U}(1)$. We write

$$P_2(x) = (x - a)(x - b) \tag{3.13}$$

Our moduli space $\mathcal{M}$ with $n = 2$ is parametrized by a and b modulo the exchange $a \leftrightarrow b$. In other words, it is the whole $\mathcal{N} = 2$ Coulomb branch.

The semiclassical limit $\Lambda \to 0$ is also trivial since $P_2(x) = (x-a)(x-b)$ is independent of Λ. Setting $P_2(x) = \det(x - \Phi_{cl})$, we find $\langle\Phi_{cl}\rangle = (a \quad 00 \quad b)$, breaking U(2) to $\mathrm{U}(1) \times \mathrm{U}(1)$.

The subspace with a massless monopole is determined by looking for points where y^2 has a double roots. This happens when

$$(a - b)^2 = \pm 8\Lambda^2. \tag{3.14}$$

These subspaces are also on the unbroken U(2) branch corresponding to $n = 1$. If we shift x such that $a = -b$ (as in SU(2)), at these points

$$P_2(x) = x^2 \mp 2\Lambda^2, \quad y^2 = x^2\left(x^2 \mp 4\Lambda^4\right). \tag{3.15}$$

Note that $P_2(x) = x^2 - 2\rho^2\Lambda^2$ with $\rho^4 = 1$; this is equal to $2\Lambda^2\rho^2\mathcal{T}_2(\frac{x}{2\rho\Lambda})$ from section 2.3. After a small perturbation causing the monopoles to condense, one of these points has confinement and one has oblique confinement.

We now break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by turning on a cubic superpotential chosen to put the system at the point (a, b). We find it through the matrix model curve

$$y_m^2 = P_2^2 - 4\Lambda^4 = ((x - a)(x - b))^2 - 4\Lambda^4 \tag{3.16}$$

from which we derive

$$\begin{aligned} W'(x) &= (x - a)(x - b) \\ f(x) &= -4\Lambda^4 \\ S &= 0. \end{aligned} \tag{3.17}$$

In this case, vanishing of S can be attributed to the discrete chiral symmetry.

If W' is given, (3.17) determines a and b modulo the exchange $a \leftrightarrow b$. Since the moduli space is obtained by dividing out by this exchange, there is in this example only one vacuum for given W. This occurs whenever all N_i are 1, as then there is no strong dynamics at low energies.

3.2 U(3)

Our next example is somewhat more interesting. It is based on the gauge group U(3).

We start by finding the subspace of the $\mathcal{N} = 2$ moduli space with one monopole. The U(3) curve

$$y^2 = \left(P_3(x) + 2\Lambda^3\right)\left(P_3(x) - 2\Lambda^3\right) \tag{3.18}$$

should have a single double root. The solution of this problem has two branches labeled by $\eta = \pm 1$ depending on which factor of (3.18) has the double root; i.e. $(s_+, s_-) = (0,1)$ or $(1,0)$. In either case, the factorization we need is

$$P_3(x) + 2\eta\Lambda^3 = (x-a)^2(x-b)\,, \tag{3.19}$$

implying also

$$P_3(x) - 2\eta\Lambda^3 = (x-a)^2(x-b) - 4\eta\Lambda^3\,. \tag{3.20}$$

As in the previous example, a and b are the parameters on the moduli space; now there is no symmetry exchanging them.

The semiclassical limit $\Lambda \to 0$ leads to $P_3(x) \to (x-a)^2(x-b)$ and therefore it corresponds to U(3) broken to U(2) $\times$ U(1). The existence of two branches — via the choice of η — has a simple explanation: the SU(2) that is unbroken classically is strongly coupled at low energies and has two supersymmetric vacua, corresponding in our problem to the two branches.

Each branch has three one-dimensional subspaces with one more massless monopole. They are determined by looking for points where y^2 has two double roots. This happens when

$$(a-b)^3 = 27\eta\Lambda^3\,. \tag{3.21}$$

Since these points have $(s_+, s_-) = (1,1)$, at these points the two branches labeled by η meet each other, and they also meet the $n = 1$ subspace of the $\mathcal{N} = 2$ moduli space. Therefore, they can be interpreted as unbroken U(3) vacua, in keeping with the discussion in section 2.4. In fact, once we impose (3.21) in the form $(a-b) = 3\rho\Lambda$ with $\rho^6 = 1$ and shift $x \to x + \frac{1}{3}(2a+b)$, P_3 becomes $x^3 - 3\rho^2\Lambda^2 x$ which is equal to $2\Lambda^3\rho^3\mathcal{T}_3(\frac{x}{2\rho\Lambda})$ from section 2.3.

We now break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by turning on a superpotential which puts the system at the point (a,b) in the $\mathcal{N} = 2$ moduli space. We find it through the matrix model curve, as in (3.5):

$$y_m^2 = (x-b)\left((x-a)^2(x-b) - 4\eta\Lambda^3\right) = ((x-a)(x-b))^2 - 4\eta\Lambda^3(x-b)\,. \tag{3.22}$$

From this we derive

$$\begin{aligned} W'(x) &= (x-a)(x-b) \\ f(x) &= -4\eta\Lambda^3(x-b) \\ S &= \eta\Lambda^3\,. \end{aligned} \tag{3.23}$$

For fixed superpotential, there are two values of (a, b). Either of them can be the double root of $P_3 + 2\eta\Lambda^2$, and the vacuum also depends on the choice of η. So for each tree level superpotential, the system has four vacua with low energy gauge group $\mathrm{U}(1)^2$. Semi-classically, they can be interpreted as follows. First, the unbroken U(2) can be associated with either one of the two different minima of the potential, spontaneously breaking the global $\mathbf{Z}_2$ symmetry. Each of these choices leads from the strong coupling SU(2) dynamics to two vacua, which differ by the sign of $S = \eta\Lambda^3$.

3.3 U(4)

We now repeat the same analysis for U(4). We look for the subspace of the $\mathcal{N} = 2$ theory with $N - n = 4 - 2 = 2$ magnetic monopoles. The two double roots of the $\mathcal{N} = 2$ curve

$$y^2 = (P_4(x)^2 + 2\Lambda^4)(P_4(x)^2 - 2\Lambda^4) \tag{3.24}$$

can be distributed between the two factors as $(s_+, s_-) = (1, 1)$, $(2, 0)$ or $(0, 2)$.

Confining branches, monopoles distributed as $(s_+, s_-) = (2, 0)$ **or** $(0, 2)$**.** With both double roots in the same factor of $P_4^2 - 4\Lambda^8$, we get two branches, labeled by the choice of $\eta = \pm 1$:

$$\begin{aligned} P_4(x) + 2\eta\Lambda^4 &= (x^2 - ax + b)^2 \\ P_4(x) - 2\eta\Lambda^4 &= (x^2 - ax + b)^2 - 4\eta\Lambda^4 \,. \end{aligned} \tag{3.25}$$

The semiclassical limit of $\Lambda \to 0$ leads to $P_4(x) \to (x^2 - ax + b)^2$, which has two double roots, so U(4) is broken to $\mathrm{U}(2) \times \mathrm{U}(2)$. That this is the only classical limit for $(s_+, s_-) = (2, 0)$ or $(0, 2)$ could be predicted from the reasoning in section 2.4.

Each branch has four singular subspaces with another massless monopole, i.e., on which $P_4(x) - 2\eta\Lambda^4$ also has a double root. This happens at the solutions of

$$\left(4b - a^2\right)^2 = 64\eta\Lambda^4. \tag{3.26}$$

These points correspond to $(s_+, s_-) = (2, 1)$ or $(1, 2)$, so they are on the branch with $n = 1$ and unbroken U(4).

As in (3.5), the matrix model curve is

$$y_m^2 = (x^2 - ax + b)^2 - 4\eta\Lambda^4 \tag{3.27}$$

from which we derive

$$\begin{aligned} W'(x) &= x^2 - ax + b \\ f(x) &= -4\eta\Lambda^4 \\ S &= 0 \,. \end{aligned} \tag{3.28}$$

We see that for each superpotential, we have two vacua (labeled by η), one on each branch. This can be understood semiclassically as follows. The low energy $\mathrm{U}(2) \times \mathrm{U}(2)$ is characterized by scales which are given in the semiclassical limit by $\Lambda_1^6 = \Lambda_2^6 \approx \frac{\Lambda^8}{a^2 - 4b}$. Gluino condensation in these groups leads to four vacua $S_1 \approx \pm\Lambda_1^3$, $S_2 \approx \pm\Lambda_1^3$. The $\mathbf{Z}_2$ exchanges

S_1 and S_2, and since it is an R-symmetry, it acts by $S_1 \leftrightarrow -S_2$. There should therefore be two $\mathbf{Z}_2$-invariant vacua, with $S_1 = -S_2$ and $S = 0$; these are evidently the vacua that we have found on the branches with $(s_+, s_-) = (2,0)$ or $(0,2)$. We will clearly have to find the two vacua with $S_1 = +S_2$ and $S \neq 0$ elsewhere.

Of the four $\mathrm{U}(2) \times \mathrm{U}(2)$ vacua, the analysis of section 2.1 shows that two are confining and two are in a Coulomb phase. In fact, the vacua that we have found are confining, because the branch that we have just described can be constructed by "multiplication by 2," of $P_2(x) = x^2 - ax + b$ as in section 2.3,

$$P_4(x) = 2\Lambda^4 \rho^2 \mathcal{T}_2\left(\frac{P_2(x)}{2\rho\Lambda^2}\right) \tag{3.29}$$

with $\rho^4 = 1$ and $\rho^2 = \eta$. This is the same as (3.25).

Coulomb branch, monopoles distributed as $(s_+, s_-) = (1,1)$**.** The other branch occurs when each of the factors $P_4(x) \pm 2\Lambda^4$ has a single double root. It is not too hard to show that, modulo the freedom to add a constant to x, this implies that

$$\begin{aligned} P_4(x) + 2\Lambda^4 &= (x-a)^2\left[(x+a)^2 + \frac{\Lambda^4}{a^3}(x+2a)\right] \\ P_4(x) - 2\Lambda^4 &= (x+a)^2\left[(x-a)^2 + \frac{\Lambda^4}{a^3}(x-2a)\right] \end{aligned} \tag{3.30}$$

for some a.

There now are two semiclassical limits with $\Lambda \to 0$:

1. Fixed a: $P_4(x) \to (x-a)^2(x+a)^2$. Therefore $\mathrm{U}(4)$ is broken to $\mathrm{U}(2) \times \mathrm{U}(2)$.

2. $\Lambda, a \to 0$ with fixed $v = \Lambda^4/a^3$: $P_4(x) \to x^3(x+v)$, showing that here $\mathrm{U}(4)$ is broken to $\mathrm{U}(3) \times \mathrm{U}(1)$.

This is the first example of our new duality: the same moduli space has two distinct semiclassical limits corresponding to different gauge groups. One can continuously deform a $\mathrm{U}(2) \times \mathrm{U}(2)$ vacuum to a $\mathrm{U}(3) \times \mathrm{U}(1)$ vacuum.

Since the numbers 1 and 3 are relatively prime, these $\mathrm{U}(3) \times \mathrm{U}(1)$ vacua are all in a Coulomb phase. This whole branch is therefore in a Coulomb phase, and the classical limit with $\mathrm{U}(2) \times \mathrm{U}(2)$ will turn out to give the two $\mathrm{U}(2) \times \mathrm{U}(2)$ vacua that did not appear earlier. The upshot will be that $\mathrm{U}(2) \times \mathrm{U}(2)$ vacua that are in a Coulomb phase can be smoothly transformed to $\mathrm{U}(3) \times \mathrm{U}(1)$ vacua, while confining $\mathrm{U}(2) \times \mathrm{U}(2)$ vacua cannot make such a transformation, since there are no confining $\mathrm{U}(3) \times \mathrm{U}(1)$ vacua.

The special points on this branch where another monopole becomes massless are at the solutions of

$$16a^8 = \Lambda^8\,. \tag{3.31}$$

Explicitly, $a = \frac{\sqrt{2}}{2}\rho\Lambda$ with $\rho^8 = 1$. Note that shifting $x \to x - \frac{\Lambda^4}{4a^3}$ and using the solutions to (3.31) we get from (3.30) that $P_4(x) = 2\Lambda^4\rho^4\mathcal{T}_4(\frac{x}{2\rho\Lambda})$. These are the same four points that we found in (3.26).

From (3.30), we find the matrix model curve, as in (3.5):

$$\begin{aligned} y_m^2 &= \left[(x+a)^2 + \frac{\Lambda^4}{a^3}(x+2a)\right]\left[(x-a)^2 + \frac{\Lambda^4}{a^3}(x-2a)\right] \\ &= \left(x^2 + x\frac{\Lambda^4}{a^3} - a^2\right)^2 - \frac{4\Lambda^4}{a}x - \frac{4\Lambda^8}{a^4}\,. \end{aligned} \tag{3.32}$$

The superpotential and $f(x)$ are therefore

$$\begin{aligned} W'(x) &= x^2 + x\frac{\Lambda^4}{a^3} - a^2 \\ f(x) &= -\frac{4\Lambda^4}{a}x - \frac{4\Lambda^8}{a^4} \\ S &= \frac{\Lambda^4}{a}\,. \end{aligned} \tag{3.33}$$

Finally, we count the number of vacua for fixed tree level superpotential. The most convenient way to proceed is to compute the discriminant of the quadratic polynomial $W' = x^2 + x(\Lambda^4/a^3) - a^2$; it is

$$\Delta = 4a^2 + \frac{\Lambda^8}{a^6}\,. \tag{3.34}$$

By shifting x, we could put the superpotential in the form (3.8). If Δ is given, then (3.34) is equivalent to an eighth order equation for a, namely

$$4a^8 - \Delta a^6 + \Lambda^8 = 0\,. \tag{3.35}$$

So there are eight possible values of a for every Δ, implying that the theory has eight vacua for each tree level superpotential. In the semiclassical limit of large Δ/Λ^2, these vacua can be understood as follows:

1. $\mathrm{U}(3)\times\mathrm{U}(1)$ leads to 3×2 vacua. The factor of 2 arises from the broken global symmetry which exchanges the two minima of the potential. Indeed, for large Δ/Λ^2, (3.35) has six roots with $a \sim (\Lambda^8/\Delta)^{1/6}$.

2. $\mathrm{U}(2)\times\mathrm{U}(2)$ leads to $2\times 2-2 = 2$ vacua. Here we do not need to include the factor of 2 which exchanges the two minima of the potential and we subtract the two confining vacua that we found on other branches. To check this prediction, we see that for large Δ/Λ^2, (3.35) has two roots with $a \sim (\Delta/4)^{1/2}$.

3.4 U(5)

To find points on the Coulomb branch for U(5) with three massless monopoles, we want polynomials P_5 such that

$$y^2 = \left(P_5(x) + 2\Lambda^5\right)\left(P_5(x) - 2\Lambda^5\right) \tag{3.36}$$

has $N - n = 5 - 2 = 3$ double zeros. This leads to two branches with $(s_+, s_-) = (2,1)$ or $(1,2)$. We will label them by $\eta = \pm 1$. $P_5 + 2\eta\Lambda^5$ should have a pair of double roots, and

$P_5 - 2\eta\Lambda^5$ should have a single double root; we use the freedom of shifting x to place this double root at the origin. Requiring the existence of the stated double roots leads to

$$\begin{aligned} P_5(x) + 2\eta\Lambda^5 &= (x^2 + ax - 2ac)^2(x+c) \\ P_5(x) - 2\eta\Lambda^5 &= x^2\left[x^3 + (2a+c)x^2 + a(a-2c)x - ac(3a+4c)\right] \end{aligned} \tag{3.37}$$

where the parameters a and c are constrained to satisfy

$$a^2c^3 = \eta\Lambda^5\,. \tag{3.38}$$

From (3.38) it is clear that there are two classical limits: as $\Lambda \to 0$, either $a \to 0$ or $c \to 0$. In the former case, $P_5(x) \to x^4(x+c)$ corresponding to $\mathrm{U}(4) \times \mathrm{U}(1)$ while in the latter, $P_5(x) \to x^3(x+a)^2$ giving $\mathrm{U}(3) \times \mathrm{U}(2)$. This is another example showing that the same branch can have different classical limits. (If we let both a and c to to zero, we will get $P_5(x) = x^5$, corresponding to a classical limit with unbroken $\mathrm{U}(5)$.)

The points with another massless monopole correspond to $(s_+, s_-) = (2,2)$. They occur when $P_5(x) - 2\eta\Lambda^5$ has two double roots. This happens when $a^2 + 11ac - c^2 = 0$ subject to the constraint (3.38). Solving for c we get,

$$c = \frac{1}{2}\left(11 + 5\epsilon\sqrt{5}\right)a \qquad \text{with} \qquad \epsilon^2 = 1 \tag{3.39}$$

and

$$a^5 = \left(-682 + 305\sqrt{5}\epsilon\right)\eta\Lambda^5 = \left(-2 + \epsilon\sqrt{5}\right)^5\eta\Lambda^5\,. \tag{3.40}$$

Equation (3.40) has 10 solutions for each η

$$a = \xi(-2 + \epsilon\sqrt{5})\Lambda\,, \qquad \xi^5 = \eta\,. \tag{3.41}$$

However, using the freedom to shift x such as to cancel the x^4 term in $P_5(x)$ (bring it to the $\mathrm{SU}(5)$ form), $x = \widetilde{x} - \frac{1}{2}a(3 + \sqrt{5}\epsilon)$, (3.37) becomes

$$\begin{aligned} P_5(x) + 2\eta\Lambda^5 &= (\widetilde{x} + 2\xi\Lambda)\left(\widetilde{x}^2 - \xi\Lambda\widetilde{x} - \xi^2\Lambda^2\right)^2 \\ P_5(x) - 2\eta\Lambda^5 &= (\widetilde{x} - 2\xi\Lambda)\left(\widetilde{x}^2 + \xi\Lambda\widetilde{x} - \xi^2\Lambda^2\right)^2 \end{aligned} \tag{3.42}$$

from which it is clear that there are only 5 different points on each branch. Note that this is equal to $P_5(x) = 2\Lambda^5\xi^5\mathcal{T}_5(\frac{x}{2\xi\Lambda})$ from section 2.3.

Why has the same P_5 appeared for two different values of the pair (a,c)? These values of (a,c) are characterized by the fact that $P_5(x) - 2\eta\Lambda^5$ has a second double root — but there are two choices of which double root is the "second" one.

Using the recipe of (3.5), we identify the matrix model curve,

$$y_m^2 = (x+c)\left[x^3 + (2a+c)x^2 + a(a-2c)x - ac(3a+4c)\right]. \tag{3.43}$$

From this we find

$$\begin{aligned} W'(x) &= x^2 + (a+c)x - ac \\ f(x) &= -4c^2a(x+a+c) \\ S &= c^2a\,. \end{aligned} \tag{3.44}$$

Finally, we count the number of vacua for fixed tree level superpotential, by the same reasoning as in previous examples. By computing the discriminant of W', we get $\Delta = a^2 + 6ac + c^2$. Using (3.38) to solve for c, we find that for fixed Δ there are 20 vacua, corresponding to the roots of

$$a^8 \left(a^2 - \Delta\right)^6 - 2a^4 \left(17a^2 + \Delta\right) \left(1153a^4 + 142a^2\Delta + \Delta^2\right) \Lambda^{10} + \Lambda^{20} = 0\,. \tag{3.45}$$

In the semiclassical limit of large Δ/Λ^2, these vacua can be understood as follows:

1. $\mathrm{U}(4) \times \mathrm{U}(1)$ leads to $4 \times 2 = 8$ vacua, with $a \sim \exp(\pm 2\pi i/8)(\Lambda^{10}/\Delta^3)^{1/4}$.
2. $\mathrm{U}(3) \times \mathrm{U}(2)$ leads to $3 \times 2 \times 2 = 12$ vacua, which are at roughly $a \sim \pm\sqrt{\Delta}$.

We have not found any confining branches for U(5) for the simple reason that 5 is a prime number, so if $5 = N_1 + N_2$, then N_1 and N_2 are relatively prime. Hence for U(5), all vacua are in a Coulomb phase.

3.5 U(6)

U(6) is the last and richest example that we will examine in detail. We look for points in the Coulomb branch of the $\mathcal{N} = 2$ theory with $N - n = 6 - 2 = 4$ double zeros. These can be distributed between the two factors of

$$y^2 = P_6(x)^2 - 4\Lambda^{12} = (P_6(x) - 2\Lambda^6)(P_6(x) + 2\Lambda^6) \tag{3.46}$$

as $(s_+, s_-) = (3,1)$, $(1,3)$ or $(2,2)$.

Confining $\mathrm{U}(2) \times \mathrm{U}(4)$ branches, monopoles distributed as $(s_+, s_-) = (1,3)$ or $(3,1)$. We first wish to consider the branches of the moduli space of U(6) with four massless magnetic monopoles distributed as $(3,1)$ and $(1,3)$ between the two factors. The solution of the factorization problem (3.2) is

$$\begin{aligned} P_6(x) + 2\eta\Lambda^6 &= \left[(x-a)^2(x-b) - 2\epsilon\Lambda^3\right]^2 \\ P_6(x) - 2\eta\Lambda^6 &= (x-a)^2(x-b)\left[(x-a)^2(x-b) - 4\epsilon\Lambda^3\right] \end{aligned} \tag{3.47}$$

with $\eta^2 = 1$ and $\epsilon^2 = \eta$.

The semiclassical limit $\Lambda \to 0$ leads to $P_6(x) \to (x-a)^4(x-b)^2$, and therefore U(6) is broken to $\mathrm{U}(4) \times \mathrm{U}(2)$. That this is the only semi-classical limit with $(s_+, s_-) = (3,1)$ or $(1,3)$ follows from the reasoning in section 2.4.

The points with another massless monopole occur when $P_6(x) - 2\eta\Lambda^6$ has two double roots. This happens when $(a-b)^6 = (3\Lambda)^6$. These special points on the $(s_+, s_-) = (3,1)$ branches have $(s_+, s_-) = (3,2)$; they also meet the $(s_+, s_-) = (2,2)$ branches we will discuss below and the U(6) branch with $n = 1$. The special points on the $(s_+, s_-) = (1,3)$ branches have $(s_+, s_-) = (2,3)$ and they meet the $(s_+, s_-) = (2,2)$ branches and the $n = 1$ branch.

Using (3.5), the matrix model curve is found from (3.47) to be

$$y_m^2 = (x-a)^2(x-b)^2 - 4\epsilon\Lambda^3(x-b)\,. \tag{3.48}$$

From this we derive

$$\begin{aligned} W'(x) &= (x-a)(x-b) \\ f(x) &= -4\epsilon\Lambda^3(x-b) \\ S &= \epsilon\Lambda^3\,. \end{aligned} \tag{3.49}$$

For fixed superpotential $\Delta = (a-b)^2$ there are 8 solutions. They correspond to the four solutions of $\epsilon^4 = 1$ for each choice of $(a-b) = \pm\sqrt{\Delta}$. In the semiclassical limit these 8 vacua can be understood as follows. $\mathrm{U}(4) \times \mathrm{U}(2)$ gives a total of $(4 \times 2) \times 2 = 16$ vacua, where 4×2 reflects the strong dynamics and a factor of 2 comes from the broken $\mathbf{Z}_2$ symmetry. But, following the analysis in section 2.1, only half of these vacua (those with b even) are confining. So we expect 8 confining vacua.

The eight vacua with $(s_+, s_-) = (3,1)$ or $(1,3)$ actually are confining, since they can be obtained by applying the "multiplication by 2" map to the polynomial $P_3(x) = (x-a)^2(x-b) - 2\rho\Lambda_0^3$ with $\rho^2 = 1$, which describes the breaking $\mathrm{U}(3) \to \mathrm{U}(2) \times \mathrm{U}(1)$ (and was presented in (3.19)). Following the general discussion of section 2.3, the scales of the theories are related by $\Lambda_0^6 = \sigma^2\Lambda^6$ with $\sigma^4 = 1$. Using this in $P_6(x) = 2\Lambda^6\sigma^2\mathcal{T}_2(\frac{P_3(x)}{2\sigma\Lambda^3})$ we recover (3.47) with $\epsilon = \rho\sigma$.

Confining $\mathrm{U}(3) \times \mathrm{U}(3)$ **branches, monopoles distributed as** $(s_+, s_-) = (2,2)$**.** We now consider the case of $(s_+, s_-) = (2,2)$. In this case, the factorization problem (3.2) has two types of solution. The first case leads to three branches parametrized by a cube root of unity ρ

$$\begin{aligned} P_6(x) + 2\Lambda^6 &= \left(x^2 + g - \rho\Lambda^2\right)^2\left(x^2 + g + 2\rho\Lambda^2\right) \\ P_6(x) - 2\Lambda^6 &= \left(x^2 + g + \rho\Lambda^2\right)^2\left(x^2 + g - 2\rho\Lambda^2\right) \\ P_6(x) &= \left(x^2 + g\right)\left[\left(x^2 + g\right)^2 - 3\rho^2\Lambda^4\right]. \end{aligned} \tag{3.50}$$

These three branches are distinct. Since P_6 depends on Λ^4 and not on Λ^2, the change $\Lambda^2 \to -\Lambda^2$ (and therefore $\Lambda^6 \to -\Lambda^6$) does not lead to more branches. Instead, there is a global symmetry $g \to -g$ which acts on each branch.

In the classical limit $\Lambda \to 0$ we have $P_6(x) \to (x^2 + g)^3$ and therefore $\mathrm{U}(6)$ is broken to $\mathrm{U}(3) \times \mathrm{U}(3)$. On each of these branches there are two "U(6) points" — points with five double roots: $g = \pm 2\rho\Lambda^2$.

We now break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by turning on a superpotential W which puts the system in the ground state labeled by g. It is determined from the matrix model curve, which is found, as in (3.5), from (3.50):

$$y_m^2 = \left(x^2 + g + 2\rho\Lambda^2\right)\left(x^2 + g - 2\rho\Lambda^2\right) = \left(x^2 + g\right)^2 - 4\rho^2\Lambda^4\,. \tag{3.51}$$

From this, we find

$$\begin{aligned} W'(x) &= x^2 + g \\ f(x) &= -4\rho^2\Lambda^4 \\ S &= 0\,. \end{aligned} \tag{3.52}$$

These expressions are consistent with the $\mathbf{Z}_2$ R-symmetry which maps $\Phi \to -\Phi$ and $S \to -S$.

From the expression $W' = x^2 + g$, it is clear that W determines g and hence P_N, so for fixed W, there is a single quantum vacuum on each branch. The three branches give a total of three vacua. Semiclassically, these can be interpreted as coming from $U(6) \to U(3) \times U(3)$, with the two gluino condensates in the two $SU(3)$ factors anti-aligned. As in the discussion after (3.28) of $U(4) \to U(2) \times U(2)$, this leads to $S = S_1 + S_2 = 0$, which respects the global $\mathbf{Z}_2$ R-symmetry of the system.

$U(3) \times U(3)$ is expected to give a total of $3 \times 3 = 9$ vacua, but according to the analysis in section 2.1, only 3 of them (those with $b = 0$) are confining. We claim that the three vacua obtained by the construction above are confining. In fact, they can be obtained by the "multiplication by 3" of $P_2(x) = x^2 + g$ which is a solution to $U(2) \to U(1) \times U(1)$. One can check that $P_6(x) = 2\Lambda^6\epsilon^3 \mathcal{T}_3(\frac{P_2(x)}{2\epsilon\Lambda^2})$ reproduces (3.50) with $\epsilon^6 = 1$ and $\epsilon^2 = \rho^2$ cubic roots of unity.

The fact that these three branches are confining, with a confinement index of 3, explains why they have no semi-classical limit with $U(4) \times U(2)$ or $U(5) \times U(1)$. There are three branches for a reason explained in section 2.1; they are distinguished by which of $W^u H$, for $u = 0, 1, 2$, has no area law.

Coulomb branch, monopoles distributed as $(s_+, s_-) = (2, 2)$. The second kind of solution of the factorization problem (3.2) in this case is

$$\begin{aligned}
P_6(x) + 2\Lambda^6 &= \left[x^2 + (h+g)x + \frac{(3h+g)(9h^3 + 15h^2 g - hg^2 + g^3)}{108h^2}\right]^2 \times \\
&\quad \times \left[x^2 - \frac{(h-g)(3h-g)^2(3h+g)}{108h^2}\right] \\
P_6(x) - 2\Lambda^6 &= \left[\left(x + \frac{2g}{3}\right)^2 + (h-g)\left(x + \frac{2g}{3}\right) + \frac{(3h-g)\left(9h^3 - 15h^2 g - hg^2 - g^3\right)}{108h^2}\right]^2 \times \\
&\quad \times \left[\left(x + \frac{2g}{3}\right)^2 - \frac{(h+g)(3h+g)^2(3h-g)}{108h^2}\right]
\end{aligned} \tag{3.53}$$

with g and h satisfying the constraint

$$g^5\left(g^2 - 9h^2\right)^2 = 27^3 h^3 \Lambda^6 . \tag{3.54}$$

We used a parametrization which makes the global symmetry $g \to -g$ combined with $\Lambda^6 \to -\Lambda^6$ manifest.

The classical limit $\Lambda \to 0$ is obtained at:

1. $g \to 0$ with finite h. Here $P_6(x) \to (x + \frac{h}{2})^5(x - \frac{h}{2})$; i.e. $U(6) \to U(5) \times U(1)$.

2. $g \to 3h$ with finite h. Here $P_6(x) \to x^2(x + 2h)^4$; i.e. $U(6) \to U(4) \times U(2)$.

3. $g \to -3h$ with finite h. Here $P_6(x) \to (x - 2h)^2 x^4$; i.e. $U(6) \to U(4) \times U(2)$.

4. $h, g \to 0$ with $v = g^2/h$ fixed. Here $P_6(x) \to (x^2 + \frac{v^2}{108})^3$; i.e. $U(6) \to U(3) \times U(3)$.

This is a richer example demonstrating that the same branch can have different classical limits. Since 1 and 5 are relatively prime, what we have found is clearly a Coulomb branch.

The U(6) points are obtained when either $P_6(x)+2\Lambda^6$ or $P_6(x)-2\Lambda^6$ acquires another double zero. This happens when

$$\frac{(h+g)(3h+g)^2(3h-g)}{108h^2}=0 \tag{3.55}$$

or the same equation with $g\to -g$. Taking into account (3.54), this happens only for

$$h=g \quad \text{with } g \text{ a solution of} \quad (2g)^6=27^3\Lambda^6 \tag{3.56}$$

or

$$h=-g \quad \text{with } g \text{ a solution of} \quad (2g)^6=-27^3\Lambda^6\,. \tag{3.57}$$

Superficially, this leads to 12 points. However (after an appropriate shift of x) one finds that at these points $P_6(x)\pm 2\Lambda^6$ depend only on g^2 and therefore there are only 6 such points. Indeed, both solutions lead to $P_6(x)=2\Lambda^6\rho^6\mathcal{T}_6(\frac{x}{2\rho\Lambda})$ with $\rho^{12}=1$. At these points, this branch meets the various confining branches that we have found.

We now break $\mathcal{N}=2$ to $\mathcal{N}=1$ by turning on a superpotential W. It is determined from the matrix model curve

$$\begin{aligned} y_m^2 &= \left[x^2-\frac{(h-g)(3h-g)^2(3h+g)}{108h^2}\right]\left[\left(x+\frac{2g}{3}\right)^2-\frac{(h+g)(3h+g)^2(3h-g)}{108h^2}\right] \\ &= \left(x^2+\frac{2}{3}gx+\frac{g^4-6g^2h^2-27h^4}{108h^2}\right)^2-\frac{4g^2(g^2-9h^2)}{81h}x+\frac{2g^2(g-3h)^3(g+3h)}{729h^2} \end{aligned} \tag{3.58}$$

and therefore

$$\begin{aligned} W'(x) &= x^2+\frac{2g}{3}x+\frac{g^4-6g^2h^2-27h^4}{108h^2}=\widetilde{x}^2+\frac{g^4-18g^2h^2-27h^4}{108h^2} \\ f(x) &= -\frac{4g^2(g^2-9h^2)}{81h}x+\frac{2g^2(g-3h)^3(g+3h)}{729h^2}=-\frac{4g^2(g^2-9h^2)}{81h}\widetilde{x}+\frac{2g^2(g^4-81h^4)}{729h^2} \\ S &= \frac{g^2(g^2-9h^2)}{81h}\,. \end{aligned} \tag{3.59}$$

We used the freedom to shift x by $x=\widetilde{x}-\frac{g}{3}$ to put the superpotential W in a canonical form $W=\frac{\widetilde{x}^3}{3}-\frac{\Delta}{4}\widetilde{x}$.

We would like to parametrize the theory in terms of Δ. Including the constraint (3.54), we have two equations with two unknowns:

$$\begin{aligned} \Delta &= \frac{27h^4-g^4+18g^2h^2}{27h^2} \\ g^5(g^2-9h^2)^2 &= 27^3h^3\Lambda^6\,. \end{aligned} \tag{3.60}$$

Using these equations we can solve for h,

$$h=\frac{4g^5(16g^{10}(g^2-\frac{9}{4}\Delta)^2-3(\frac{9}{4}\Delta)(3\Lambda)^{12})}{9(3\Lambda)^6(32g^{10}(g^2-\frac{3}{4}\Delta)+3(3\Lambda)^{12})} \tag{3.61}$$

and find a single equation for a single unknown g,

$$g^{16}\left(g^2-\frac{9}{4}\Delta\right)^4=(3\Lambda)^{12}\left(\frac{9}{8}g^{12}-\frac{3}{8}\left(\frac{9}{4}\Delta\right)^2 g^8+\frac{1}{4}\left(\frac{9}{4}\Delta\right)^3 g^6+\frac{27}{256}(3\Lambda)^{12}\right). \qquad (3.62)$$

In deriving this equation we assumed that $g\neq 0$.

Since the final constraint (3.62) is of degree 24, for fixed superpotential (fixed Δ) there are 24 vacua. These can be interpreted as follows:

1. $\mathrm{U}(1)\times\mathrm{U}(5)$ vacua. Their number is $1\times 5\times 2=10$ (the factor of 2 is associated with the global $\mathbf{Z}_2$ symmetry of the theory).

2. $\mathrm{U}(2)\times\mathrm{U}(4)$ vacua. Their number is $2\times 4\times 2-8=8$ (we subtracted the 8 confining vacua).

3. $\mathrm{U}(3)\times\mathrm{U}(3)$ vacua. Their number is $3\times 3-3=6$ (we subtracted the 3 confining vacua, and we did not multiply by 2 because for $\mathrm{U}(3)\times\mathrm{U}(3)$ the $\mathbf{Z}_2$ symmetry is not broken).

It is interesting to contrast this situation with the classical theory ($\Lambda\to 0$). Here we find 3 solutions: $g=0,\pm\frac{3}{2}\sqrt{\Delta}$. The corresponding values of h depend on the details of the limit. The two solutions with $g=\pm\frac{3}{2}\sqrt{\Delta}$ correspond to $\mathrm{U}(2)\times\mathrm{U}(4)$ vacua (compare with the analysis of the semiclassical limits above). Each of them splits to 4 vacua in the quantum theory. The solution $g=0$ corresponds to $\mathrm{U}(1)\times\mathrm{U}(5)$ and to $\mathrm{U}(3)\times\mathrm{U}(3)$. This point splits in the quantum theory to 16 vacua (compare with the analysis of the semiclassical limits above).

3.6 Examples with $s_-=0$

This completes what we will say by way of systematic analysis for relatively small N. In this subsection, we study another kind of example in which the factorization problem is easily solved explicitly. This is the case that all the double roots are in one factor of the $\mathcal{N}=2$ curve — for example, $s_-=0$ and therefore $s_+=N-n$. There is an analogous discussion for $s_+=0$. The condition $0\leq s_+\leq N/2$ shows that this is possible only when $n\leq N\leq 2n$. The factorization problem is solved by

$$\begin{aligned} P_N(x)+2\Lambda^N &= H_{s_+}^2(x)R_{N-2s_+}(x)\\ P_N(x)-2\Lambda^N &= H_{s_+}^2(x)R_{N-2s_+}(x)-4\Lambda^N \end{aligned} \qquad (3.63)$$

with arbitrary coefficients in $H_{s_+}^2(x)$ and $R_{N-2s_+}(x)$.

From (3.63), we easily find the semiclassical limit

$$P_N(x)\to H_{s_+}^2(x)R_{N-2s_+}(x)\,. \qquad (3.64)$$

We see that $\mathrm{U}(N)$ is broken to $\mathrm{U}(2)^{s_+}\times\mathrm{U}(1)^{N-2s_+}$.

From (3.63) the matrix model curve is

$$y_m^2=R_{N-2s_+}(x)\left[H_{s_+}^2(x)R_{N-2s_+}(x)-4\Lambda^N\right]=\left[H_{s_+}(x)R_{N-2s_+}(x)\right]^2-4\Lambda^N R_{N-2s_+}(x). \qquad (3.65)$$

If $n+1 \leq N$, the degree of $R_{N-2s_+}(x)$ is $N-2s_+ = 2n-N \leq n-1$, and we immediately find the superpotential and $f(x)$:

$$\begin{aligned} W'(x) &= H_{s_+}(x) R_{N-2s_+}(x) \\ f(x) &= -4\Lambda^N R_{N-2s_+}(x) \end{aligned} \tag{3.66}$$

If $n+2 \leq N$, the degree of $f(x)$ is smaller than $n-1$ and we conclude that $S=0$.

We have seen an example of this when we studied $n=2$ with $N=4$ above with $(s_+, s_-) = (2,0)$. As a more interesting example, consider $N=5$ with $(s_+, s_-) = (2,0)$ and therefore $n=3$. Following the general solution (3.63), we find

$$\begin{aligned} P_5(x) + 2\Lambda^5 &= (x^2+ax+b)^2(x+c) \\ P_5(x) - 2\Lambda^N &= (x^2+ax+b)^2(x+c) - 4\Lambda^5 \,. \end{aligned} \tag{3.67}$$

In the semiclassical limit, $P_5(x) \to (x^2+ax+b)^2(x+c)$, and therefore U(5) is broken to $\mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(1)$.

From (3.65), (3.66), we deduce

$$\begin{aligned} y_m^2 &= \left[(x^2+ax+b)(x+c)\right]^2 - 4\Lambda^5(x+c) \\ W'(x) &= (x^2+ax+b)(x+c) \\ f(x) &= -4\Lambda^5(x+c) \\ S &= 0\,. \end{aligned} \tag{3.68}$$

The theory has a quartic superpotential, and therefore there is no global symmetry. Yet we still find $S=0$. This is a special case of a more general phenomenon we will see in the next section.

3.7 Interpolation and duality

As discussed in section 2.4, a given branch, Coulomb or confining, can have different classical limits. We have given explicit examples of the interpolation between classical limits $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ and $\mathrm{U}(\widetilde{N}_1) \times \mathrm{U}(\widetilde{N}_2)$. Since the N_i are certain periods of the one-form T, such an interpolation must involve a rearrangement of the compact one-cycles on the Riemann surface. With some numerical work, we have determined how this happens in all the examples of section 3. Here we will simply illustrate the result qualitatively, without attempting any proofs. In the process, we will see that the two descriptions are related to each other through an electric-magnetic duality.

For a general breaking pattern $\mathrm{U}(N) \to \prod_{i=1}^n \mathrm{U}(N_i)$, there are $2n-1$ compact cycles, as sketched (for $n=3$) in figures 1 and 2. There are n cycles $A_1, \ldots, A_n$ that in a weak coupling limit surround a pair of roots, and $n-1$ cycles C_i that connect neighboring pairs. The intersection numbers are $A_i \cap A_j = C_i \cap C_j = 0$, $A_i \cap C_j = \delta_{i,j+1} - \delta_{ij}$. In particular, the sum $v = A_1 + A_2 + \ldots + A_n$ is a null vector. In fact, it is clear from figure 1 that this cycle is homologous to a cycle at infinity that winds once around the whole x-plane; this cycle will not be affected by the motion of the roots. So the null vector v will be invariant under the monodromy. If the action of the monodromies on A_i and C_j, $i,j<n$, is known, then the behavior of A_n is determined using the fact that v is invariant.

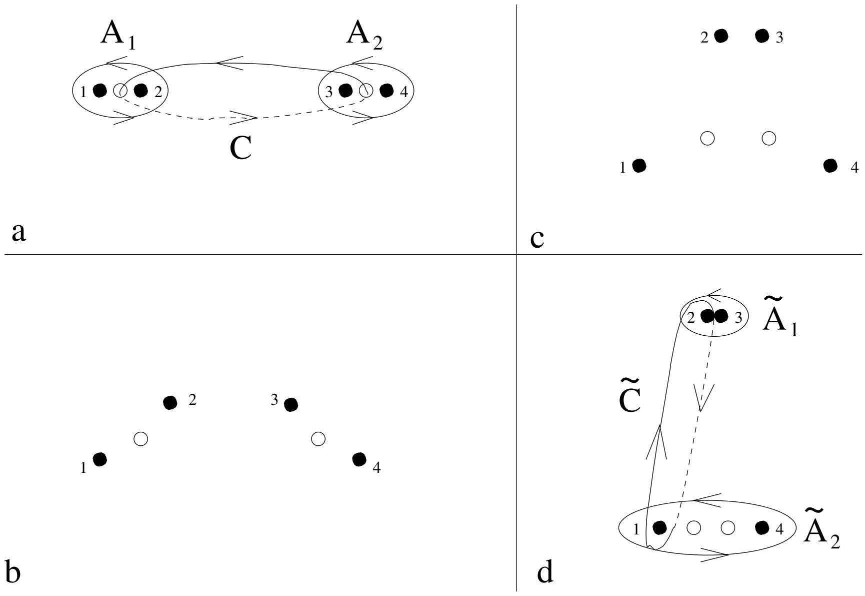

Figure 4: Interpolation between classical limits in the Coulomb branch of U(4). The polynomial $P_4^2(x) - 4\Lambda^8$ has four single roots (labeled 1,2,3 and 4) and two double roots; single and double roots are depicted by black and white circles respectively. (*a*) U(2) × U(2) semiclassical limit. Natural basis of compact cycles A_1, A_2 and C. (*b*) Semiclassical regime of U(2) × U(2) and transition to strong coupling of U(1) × U(3). (*c*) Strong coupling of U(2) × U(2) and transition to semiclassical regime of U(1) × U(3). (*d*) U(1) × U(3) semiclassical limit. Natural basis of compact cyles $\widetilde{A}_1$, $\widetilde{A}_2$ and $\widetilde{C}$.

To illustrate, we will examine examples with $n = 2$. In this case, there are just three cycles, A_1, A_2, and $C = C_1$. If we know what happens to A_1 and C, the behavior of A_2 is known. Given the intersection relations of A_1 and C, the monodromies must act on them by an $\mathrm{SL}(2, \mathbf{Z})$ transformations plus possible addition of a multiple of the null vector v. (The null vector is uniquely determined by the fact that the $\widetilde{N}_i$ are non-negative.)

Now we will sketch the mechanism of interpolation we found to operate in all examples. These branches all have complex dimension two, but one complex parameter is associated with the freedom of shifting x in $P_N(x)$. This parameter can be removed, leaving only a space with complex dimension one. We want to discuss how the zeroes of $P_N(x)^2 - 4\Lambda^{2N}$ move and the cycles evolve as one interpolates from a classical limit with unbroken $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ to a classical limit with $\mathrm{U}(\widetilde{N}_1) \times \mathrm{U}(\widetilde{N}_2)$.

We start in the semiclassical limit with the group $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$. As discussed in section 2, the polynomial $P_N(x)^2 - 4\Lambda^{2N}$ is given in this limit in terms of the Chebyshev polynomials of the two factors $\mathrm{U}(N_1)$ and $\mathrm{U}(N_2)$. Each factor leads to $N_i - 1$ double roots and two single roots. This is depicted in figure 4*a* for $N_1 = N_2 = 2$.

As we vary the superpotential to go to a strongly coupled region, the two groups mix (figures 4*b* and 4*c*) and the zeroes then separate into two new groups (figure 4*d*). One group has two single roots and $\widetilde{N}_1$ double roots, and the other has two single roots and $\widetilde{N}_2$ double roots.

It is easy to see the way the cycles are transformed. Following the steps from figure $4a$ to figure $4d$, A_1 evolves into $-\widetilde{C}$ and C evolves into $\widetilde{A}_1$. We can describe this by the monodromy matrix

$$\begin{pmatrix} \widetilde{A}_1 \\ \widetilde{C} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ C \end{pmatrix}. \tag{3.69}$$

In this example, with this choice of cycles, the monodromy is in $\mathrm{SL}(2, \mathbf{Z})$, with no addition of a null vector. In fact, the transformation matrix in (3.69) is the S generator of $\mathrm{SL}(2, \mathbf{Z})$, revealing the electric-magnetic nature of the duality. Since $A_1 + A_2 = \widetilde{A}_1 + \widetilde{A}_2$, it follows from the above that

$$\widetilde{A}_2 = A_1 + A_2 - C\,. \tag{3.70}$$

Since $N_i = \oint_{A_i} T$ and $c = \oint_C T$, we have

$$\begin{aligned} \widetilde{N}_1 &= c \\ \widetilde{N}_2 &= N_1 + N_2 - c \\ \widetilde{c} &= -N_1\,. \end{aligned} \tag{3.71}$$

For example, if we start with the Coulomb vacuum $N_1 = N_2 = 2$, $c = 1$, then we get $\widetilde{N}_1 = 1$, $\widetilde{N}_2 = 3$, $\widetilde{c} = -2$. This is the case depicted in figure 4. For producing such an interpolation, we used $P_4(x)$ in (3.30). It is convenient to introduce a dimensionless parameter $\frac{\Lambda}{a}$. The figures, from $4a$ to $4d$, show the roots of $P_4(x)^2 - 4\Lambda^8$ as the parameter changes from 0 to ∞. Other examples from section 3 can be described similarly.

Let us now interpret this in the $\mathcal{N} = 1$ theory, where superpotentials of the form (2.13) describe the low energy dynamics. To construct the $\mathcal{N} = 1$ effective superpotential for the gluino bilinears S_i, we need the periods of the differential $R(x)$ as well as those of T. We have as in [4] and eq. (2.8)

$$\begin{aligned} S_i &= \oint_{A_i} R\,, \\ \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial S_i} &= \oint_{B_i} R\,. \end{aligned} \tag{3.72}$$

To evaluate these expressions, we need the transformation laws of the noncompact cycles B_i. These are uniquely determined modulo possible addition of a null vector by the intersections with the compact cycles. Since the monodromy must also preserve $B_1 \cap B_2 = 0$, it is fixed modulo addition of a common null vector to B_1 and B_2, and this is irrelevant in the sense that it is equivalent to a 2π shift in the theta angle. So we get

$$\begin{aligned} \widetilde{B}_1 &= A_1 + B_2 \\ \widetilde{B}_2 &= B_2\,. \end{aligned} \tag{3.73}$$

Consequently, the transformation laws are

$$\widetilde{S}_1 = -\frac{1}{2\pi i}\left(\frac{\partial \mathcal{F}}{\partial S_1} - \frac{\partial \mathcal{F}}{\partial S_2}\right)$$

$$
\begin{aligned}
\widetilde{S}_2 &= S_1 + S_2 + \frac{1}{2\pi i}\left(\frac{\partial \mathcal{F}}{\partial S_1} - \frac{\partial \mathcal{F}}{\partial S_2}\right) \\
\frac{1}{2\pi i}\frac{\partial \widetilde{\mathcal{F}}}{\partial \widetilde{S}_1} &= S_1 + \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial S_2} \\
\frac{1}{2\pi i}\frac{\partial \widetilde{\mathcal{F}}}{\partial \widetilde{S}_2} &= \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial S_2}.
\end{aligned} \tag{3.74}
$$

4. Equations characterizing components of moduli space

With the exception of section 3.6, we have discussed so far examples in which the rank n of the low energy gauge group is small and the number $w = N - n$ of condensed monopoles is correspondingly large. Here we will concentrate on examples with relatively small w, and correspondingly a large rank of the low energy gauge group $\mathrm{U}(1)^n$.

If $\mathrm{U}(N)$ is broken classically to $\prod_i \mathrm{U}(N_i)$, then $w = \sum_i (N_i - 1)$. This implies that if $w < N/2$, a typical range in the following discussion, then some of the N_i are 1. This means that all vacua are Coulomb vacua and confinement is not relevant in distinguishing different branches. Nevertheless, there are different branches of the moduli space of vacua, for a reason explained in section 2.4. The different branches correspond to decompositions $w = s_+ + s_-$, where s_+ and s_- are, respectively, the number of double roots of the two polynomials $P_N(x) + 2\Lambda^{2N}$ and $P_N(x) - 2\Lambda^{2N}$.

Not having confinement as a useful order parameter and with only a very limited role for discrete symmetries, there appear to be no conventional order parameters that explain the existence of $w + 1$ branches labeled by s_+. Instead, as sketched in the introduction, we will seek here to characterize each branch by describing holomorphic functions of chiral operators that have a vanishing expectation value on a given branch. The functions with this property will depend on s_+ and s_-.

Certainly, there is no loss of essential generality in focusing on the case $s_+ \geq s_-$. If $s_+ = s_-$ or $s_+ = s_- + 1$, then the analysis below does not reveal any unusual equations obeyed on a given branch. For each w, precisely one of these possibilities is realized, and we know of no unusual equations obeyed on this branch. We will find such equations in all the other cases

$$
s_+ \geq s_- + 2\,, \tag{4.1}
$$

and these relations will depend on s_-.

To make explicit the double roots, we write

$$
\begin{aligned}
P_N + 2\Lambda^{2N} &= H_{s_+}^2 R_{N-2s_+} \\
P_N - 2\Lambda^{2N} &= \widetilde{H}_{s_-}^2 \widetilde{R}_{N-2s_-}\,.
\end{aligned} \tag{4.2}
$$

The curve of the $\mathcal{N} = 2$ theory is

$$
y^2 = \left(P_N + 2\Lambda^{2N}\right)\left(P_N - 2\Lambda^{2N}\right) = \left(\widetilde{H}_{s_-} H_{s_+}\right)^2 R_{N-2s_+} \widetilde{R}_{N-2s_-}\,. \tag{4.3}
$$

However, in the matrix model we remove the quadratic factor $(\widetilde{H}_{s_-}H_{s_+})^2$, setting $y_m = y/\widetilde{H}_{s_-}H_{s_+}$, so

$$y_m^2 = R_{N-2s_+}\widetilde{R}_{N-2s_-}\,. \tag{4.4}$$

We write this

$$y_m^2 = \frac{R_{N-2s_+}(H_{s_+}^2 R_{N-2s_+} - 4\Lambda^{2N})}{\widetilde{H}_{s_-}^2} = \left(\frac{H_{s_+}R_{N-2s_+}}{\widetilde{H}_{s_-}}\right)^2 \left(1 - \frac{4\Lambda^{2N}}{H_{s_+}^2 R_{N-2s_+}}\right). \tag{4.5}$$

As shown in [17], the order parameters $t_r = -(1/32\pi^2)\,\mathrm{Tr}\,\Phi^r W_\alpha^2$ are

$$t_r = \oint_\infty dx\, x^r y_m = \oint_\infty dx\, x^r \frac{H_{s_+}R_{N-2s_+}}{\widetilde{H}_{s_-}}\sqrt{1 - \frac{4\Lambda^{2N}}{H_{s_+}^2 R_{N-2s_+}}}\,. \tag{4.6}$$

Here the contour is over a large circle near infinity in the complex x-plane (and we recall that a factor of $1/2\pi i$ is included in the definition of the symbol $\oint$). A standard argument shows that the factor $\sqrt{1 - 4\Lambda^{2N}/H_{s_+}^2 R_{N-2s_+}}$ can be replaced by 1 if

$$r \le s_+ + s_- - 2 = w - 2 \tag{4.7}$$

as then the terms obtained by expanding the square root vanish too rapidly at infinity to contribute. The number of values of r for which this inequality is obeyed is $s_+ + s_- - 1$; given our assumption (4.1), this is at least $2s_- + 1$.

For r such that (4.7) is obeyed, we have simply

$$t_r = \oint_\infty dx\, x^r\, \frac{H_{s_+}R_{N-2s_+}}{\widetilde{H}_{s_-}}\,. \tag{4.8}$$

The rational function $H_{s_+}R_{N-2s_+}/\widetilde{H}_{s_-}$ has poles only at the s_- zeroes of $\widetilde{H}_{s_-}$, so it has the form

$$\frac{H_{s_+}R_{N-2s_+}}{\widetilde{H}_{s_-}} = \sum_{i=1}^{s_-} \frac{b_i}{x - a_i} + \text{polynomial} \tag{4.9}$$

for some complex numbers a_i, b_j. Now we evaluate (4.8) to give

$$t_r = \sum_{i=1}^{s_-} b_i a_i^r\,, \tag{4.10}$$

valid for $r \le w - 2$.

This equation expresses the $s_+ + s_- - 1$ quantities t_r, $0 \le r \le s_+ + s_- - 2$, in terms of the $2s_-$ quantities a_i and b_j. So the t_r cannot be independent; they will obey at least $s_+ - s_- - 2$ independent algebraic equations. Before working out these equations in general, let us consider the first few cases. If $s_- = 0$, we have simply $t_r = 0$, $0 \le r \le s_+ + s_- - 2$. If $s_- = 1$, we have $t_r = ba^r$ for some a and b, whence $t_i t_j = t_k t_l$ whenever $i + j = k + l$ (and $i, j, k, l \le w - 2$). The first such relation (and the only example if $s_- = 1$ and

$s_+ = s_- + 2 = 3$) is the equation $t_0 t_2 - t_1^2 = 0$. We can write this equation as the vanishing of the determinant of the 2×2 matrix

$$M_2 = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix}. \tag{4.11}$$

One way to prove that $\det M_2 = 0$ for $s_- = 1$ is to write

$$M_2 = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix}, \tag{4.12}$$

where we have written a formula for M_2 that is valid if $s_- = 2$. But if we set $s_- = 1$, we should take $b_2 = a_2 = 0$ (and then $a_1 = a, b_1 = b$) and clearly $\det M_2 = 0$ since the second factor in M_2 has vanishing determinant.

In general, for any n, we set M_n to be the matrix whose i, j matrix element is t_{i+j-2}. For example, the next case is

$$M_3 = \begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \end{pmatrix}. \tag{4.13}$$

If $s_- = n$, M_n can be written as a product of three matrices, namely

$$M_n = A_n B_n A_n^T \tag{4.14}$$

with

$$A_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & & \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{pmatrix}, \tag{4.15}$$

and B_n a diagonal matrix with eigenvalues $b_1, b_2, \dots, b_n$. (Thus, the i, j matrix element of A_n is a_j^{i-1}.) A_n is a Vandermonde matrix, whose determinant is

$$\pm \prod_{i<j} (a_i - a_j). \tag{4.16}$$

The determinant of M_n is hence

$$\det M_n = \prod_{i<j} (a_i - a_j)^2 \prod_k b_k \tag{4.17}$$

and in particular is generically nonzero for $s_- = n$.

$\det M_n$ depends on the t_r with $r \leq 2n$. Since our derivation has assumed that $r \leq w-2$, the maximum value of n that we should consider is

$$n = \left[\frac{1}{2} w - 1\right], \tag{4.18}$$

where for any real x, $[x]$ denotes the greatest integer less than or equal to x. In what follows we set n to this value. If $s_- < n$, the formula (4.14) for M_n remains valid, but we should set all but s_- of the b_i (and a_i) to zero. Hence the matrix B_n, and consequently also M_n, has rank s_-. It follows that the rank $s_- + 1$ minors of M_n (the determinants of $(s_- + 1) \times (s_- + 1)$ matrices obtained by omitting some rows and columns from M_n) vanish. These are our relations.

Each minor is a homogeneous polynomial in the t_i of degree $s_- + 1$. For example, for $s_- = 0$, the rank $s_- + 1$ minors are simply the t_i themselves, and for $s_- = 1$, they are the quadratic functions $t_i t_j - t_k t_l$ (for $i + j = k + l$) whose vanishing we found earlier for $s_- = 1$.

Since M_n is an $(n + 1) \times (n + 1)$ matrix, it has minors of all ranks up to $n + 1$ (the rank $n + 1$ minor being simply $\det M_n$). The relations we have found are hence non-trivial whenever $n \geq s_-$, or equivalently whenver $s_+ \geq s_- + 2$. For $s_+ = s_- + 2$, we have $n = s_-$, and the unique relation of this type is $\det M_{s_-} = 0$.

For any given s_-, the rank s_- minors do not generically vanish, since one of them (the determinant of the upper left $s_- \times s_-$ submatrix of M_n) is $\det M_{s_- - 1}$, which is evaluated by setting $n = s_-$ in (4.17). So a branch with given s_-, and $s_+ \geq s_- + 2$, is characterized by saying that the rank s_- minors of M_n are generically nonvanishing, while the rank $s_- + 1$ minors vanish identically. We have accomplished our goal of finding holomorphic functions of chiral order parameters whose vanishing on some branches and not others distinguishes the different branches, at least modulo the exchange $s_+ \leftrightarrow s_-$.

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A. Generalization to superpotentials of arbitrary degree

In section 2.2 we restricted the degree of the superpotential to be less than $N+1$. The main reason was that only the first N coordinates $u_i's$ are independent. In this section we will generalize the discussion by allowing a superpotential of any degree. The idea is to regard all u_i's as independent coordinates on a larger space subject to constraints. Classically, a simple way of obtaining the constraints is by looking at the generating function of $u_i's$,

$$\mathrm{Tr}\,\frac{1}{x - \Phi_{cl}} = \sum_{l=0}^{\infty} \frac{\mathrm{Tr}\ \Phi_{cl}^l}{x^{l+1}} = \frac{N}{x} + \sum_{l=1}^{\infty} \frac{l u_l}{x^{l+1}}\,. \tag{A.1}$$

Notice that this object is also written in terms of $P_N(x) = \det(x - \Phi_{cl})$ which is a function only of $u_i's$ for $i = 1, \ldots, N$ as follows,

$$\mathrm{Tr}\,\frac{1}{x - \Phi} = \frac{P'_N(x)}{P_N(x)} = \frac{d}{dx} \log P(x)\,. \tag{A.2}$$

Integrating (A.2) and (A.1) with respect to x, we get,

$$P(x) = x^N \exp\left(- \sum_{i=1}^{\infty} \frac{u_i}{x^i}\right) \tag{A.3}$$

where the integration constant was fixed by matching the x^N terms.

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The constraints we are after come from imposing (A.3) i.e, the Laurent series has to terminate at order x^0. Since the coefficient of x^{-l} only depends on u_i with $i \leq l$ and is linear in u_l, imposing that to vanish gives a system of equations for u_l with $l \geq N+1$ in triangular form that can be solved in terms of the first N coordinates.

In writing the effective superpotential (2.26) one has to use $\langle \mathrm{Tr}\ \Phi^l \rangle$ instead of $\mathrm{Tr}\ \Phi_{cl}^l$. In the first section we did not make any distinction since $\langle \mathrm{Tr}\ \Phi^l \rangle = \mathrm{Tr}\ \Phi_{cl}^l$ for $l \leq 2N-1$. For $l \geq 2N$ the relation receives instanton corrections. This implies that a modification to the constraints is needed. We now turn to this issue.

Consider the quantum mechanical analog of (A.1),

$$\left\langle \mathrm{Tr}\,\frac{1}{x-\Phi} \right\rangle = \frac{N}{x} + \sum_{l=1}^{\infty} \frac{lU_l}{x^{l+1}} \tag{A.4}$$

where we have denoted $\frac{1}{l}\langle \mathrm{Tr}\ \Phi^l \rangle$ by U_l.

The full quantum mechanical analog of (A.2) is,

$$\left\langle \mathrm{Tr}\,\frac{1}{x-\Phi} \right\rangle = \frac{P_N'(x)}{\sqrt{P_N^2(x) - 4\Lambda^{2N}}} = \frac{d}{dx}\log\left(P_N(x) + \sqrt{P_N^2(x) - 4\Lambda^{2N}}\right). \tag{A.5}$$

Integrating with respect to x both (A.4) and (A.5) we get,

$$\frac{1}{2}\left(P_N(x) + \sqrt{P_N^2(x) - 4\Lambda^{2N}}\right) = x^N \exp\left(-\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right) \tag{A.6}$$

where the constant of integration was determined to be $\frac{1}{2}$ by taking Λ to zero.

Finally we can find the analog of (A.3) by solving for $P_N(x)$ in (A.6),

$$P_N(x) = x^N \exp\left(-\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right) + \frac{\Lambda^{2N}}{x^N}\exp\left(\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right). \tag{A.7}$$

Now we are ready to write the generalization of (2.26) for a superpotential of any degree.

$$\begin{aligned} W_{\mathrm{eff}} = & \sum_{r=0}^{k} g_r U_{r+1} + \oint V_{k-N}(x)\left(x^N \exp\left(-\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right) + \frac{\Lambda^{2N}}{x^N}\exp\left(\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right)\right) dx + \\ & + \sum_{i=1}^{N-n}\left(L_i\left(\oint \frac{P_N(x)}{x-p_i}dx - 2\epsilon_i\Lambda^N\right) + B_i \oint \frac{P_N(x)}{(x-p_i)^2}dx\right) \end{aligned} \tag{A.8}$$

where $V_{k-N}(x)$ is a polynomial of degree $k-N$ whose coefficients are to be thought of as Lagrange multipliers imposing constraints determining U_l for $l = N+1, \ldots, k+1$ in terms of U_l with $l = 1, \ldots, N$ and Λ.

The next step is to follow the computations in section 2. The derivative of W_{eff} with respect to U_{r+1} is

$$\begin{aligned} g_r + \oint \frac{V_{k-N}(x)}{x^{r+1}}\left(-x^N \exp\left(-\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right) + \frac{\Lambda^{2N}}{x^N}\exp\left(\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right)\right) dx + \\ + \oint \frac{Q_{N-n-1}(x)}{H_{N-n}(x)}\frac{\partial P_N(x)}{\partial U_{r+1}}dx = 0\,. \end{aligned} \tag{A.9}$$

Using (2.31) and the fact that in the relevant range of r we have $U_r = u_r$,

$$\frac{\partial P_N(x)}{\partial U_{r+1}} = \begin{cases} -\frac{P_N(x)}{x^{r+1}} & r+1 \leq N \\ 0 & r+1 > N \end{cases} .$$

However, inside the integral we can write $\frac{\partial P_N(x)}{\partial u_{r+1}} = -\frac{P_N(x)}{x^{r+1}}$ for any r since for $r+1 > N$ the integral vanishes.

In order to simplify (A.9) note that after imposing the $V_{k-N}(x)$ constraints,

$$x^N \exp\left(-\sum_{i=1}^{\infty} \frac{U_i}{x^i}\right) = \frac{1}{2}\left(P_N(x) + \sqrt{P_N^2(x) - 4\Lambda^{2N}}\right) + \mathcal{O}(x^{k+1}) .$$

It is easy to see that the $\mathcal{O}(x^{k+1})$ terms do not contribute to the integrals. Therefore, we can write the integral in (A.9) that contains $V_{k-N}(x)$ as,

$$\oint \frac{V_{k-N}(x)}{x^{r+1}} \left(-\frac{1}{2}\left(P_N(x) + \sqrt{P_N^2(x) - 4\Lambda^{2N}}\right) + \frac{2\Lambda^{2N}}{P_N(x)+\sqrt{P_N^2(x)-4\Lambda^{2N}}}\right) dx =$$
$$- \oint \frac{V_{k-N}(x)}{x^{r+1}} \sqrt{P_N^2(x) - 4\Lambda^{2N}} dx.$$

Using these results we can multiply (A.9) by z^r and sum over r from zero to k to get,

$$W'(z) = \oint \frac{V_{k-N}(x)}{(x-z)} \sqrt{P_N^2(x) - 4\Lambda^{2N}} dx + \oint \frac{Q_{N-n-1}(x)}{H_{N-n}(x)} \frac{P_N(x)}{(x-z)} dx .$$

Finally, using (2.24) we can replace $\sqrt{P_N^2(x) - 4\Lambda^{2N}}$ by $\sqrt{F_{2n}(x)} H_{N-n}(x)$ in the first integral and $P_N(x)$ by $\sqrt{F_{2n}(x)} H_{N-n}(x) + \mathcal{O}(x^{-N})$ in the second. As in section 2.2, the $\mathcal{O}(x^{-N})$ does not contribute to the integral. Note that had we not included instanton corrections in the constraints, we would have obtained $P_N(x)$ instead of $\sqrt{P_N^2(x) - 4\Lambda^{2N}}$ in the first integral and the $\mathcal{O}(x^{-N})$ could not have been dropped for $k \geq 2N$.

The final result is thus

$$W'(z) = \oint \sqrt{F_{2n}(x)} \left(V_{k-N}(x) H_{N-n}(x) + Q_{N-n-1}(x)\right) \frac{dx}{(x-z)} . \tag{A.10}$$

This agrees with the matrix model equation of motion

$$W'(z) = \oint \frac{y_m(x) dx}{x - z} \tag{A.11}$$

if the matrix model curve is

$$y_m^2(x) = F_{2n}(x) \widetilde{Q}_{k-n}^2(x) .$$

where $\widetilde{Q}_{k-n}(x) = V_{k-N}(x) H_{N-n}(x) + Q_{N-n-1}(x)$. Moreover, (A.10) implies that y_m^2 is known up to a polynomial $f_{k-1}(x)$ of degree $k-1$, i.e.

$$y_m^2 = F_{2n}(x) \widetilde{Q}_{k-n}^2(x) = W_k'^2(x) + f_{k-1}(x),$$

providing the generalization of (2.25) and the result (2.41) discussed in section 2.2.

Let us consider some special cases:

1. No massless monopoles $n = N$: Then $\widetilde{Q}_{k-N}(x) = V_{k-N}(x)$.

2. Degree of superpotential equal to $N+1$, i.e. $k = N$: $\widetilde{Q}_{k-n}(x) = V_0 H_{N-n}(x) + Q_{N-n-1}(x)$. In particular, for $n = N$, i.e. U(N) completely broken to U(1)^N, $\widetilde{Q}_0$ is a constant and $y_m^2(x) = F_{2N}(x) = P_N^2(x) - 4\Lambda^{2N}$.

B. Proof of the generalized Konishi anomaly from strong coupling analysis

In this appendix, using the results from section 2.2 we show that the generalized Konishi anomaly equation

$$\left\langle \mathrm{Tr}\, \frac{W'(\Phi)}{z-\Phi} \right\rangle = 2R(z) \left\langle \mathrm{Tr}\, \frac{1}{z-\Phi} \right\rangle \tag{B.1}$$

follows from the effective superpotential (2.26). For simplicity we will assume that the degree of W is less than $2N+1$ such that $\langle \mathrm{Tr}\, W'(\Phi)\rangle = \mathrm{Tr}\, W'(\Phi_{cl})$.

Instead of viewing u_r as the coordinates on the $\mathcal{N} = 2$ Coulomb branch, we can use Φ_{cl} and mod out by U(N). This is valid except at points where some of the eigenvalues of Φ_{cl} coincide. Then, instead of varying $W_{\rm eff}$ with respect to u_r as in (2.33), we vary with respect to ϕ_I, the eigenvalues of Φ_{cl} (recall that $P_N(x) = \prod_I (x - \phi_I)$), to get

$$W'(\phi_I) = \sum_{i=1}^{N-n} L_i \oint \frac{P_N(x)}{(x-\phi_I)(x-p_i)} dx \tag{B.2}$$

(We have used the result $B_i = 0$ from section 2.2.) From this we derive that with z outside the contour of integration

$$\begin{aligned} \mathrm{Tr}\, \frac{W'(\Phi_{cl})}{z-\Phi_{cl}} &= \sum_{I=1}^{N} \sum_{i=1}^{N-n} L_i \oint \frac{P_N(x)}{(z-\phi_I)(x-\phi_I)(x-p_i)} dx \\ &= \sum_{I=1}^{N} \sum_{i=1}^{N-n} L_i \oint \frac{P_N(x)}{(z-x)(x-p_i)} \left(\frac{1}{x-\phi_I} - \frac{1}{z-\phi_I} \right) dx \\ &= \sum_{i=1}^{N-n} L_i \oint \frac{P_N(x)}{(z-x)(x-p_i)} \left(\mathrm{Tr}\, \frac{1}{x-\Phi_{cl}} - \mathrm{Tr}\, \frac{1}{z-\Phi_{cl}} \right) dx\,. \end{aligned} \tag{B.3}$$

Using (2.27) $P_N(p_i)\, \mathrm{Tr}\, \frac{1}{p_i - \Phi_{cl}} = 0$, and the fact that $P_N(\phi_I) = 0$ the first term does not contribute. The second term can be simplified using (2.35)

$$\begin{aligned} \mathrm{Tr}\, \frac{W'(\Phi_{cl})}{z-\Phi_{cl}} &= -\,\mathrm{Tr}\, \frac{1}{z-\Phi_{cl}} \sum_{i=1}^{N-n} L_i \oint \frac{P_N(x)}{(z-x)(x-p_i)} dx \\ &= -\,\mathrm{Tr}\, \frac{1}{z-\Phi_{cl}} \oint \frac{P_N(x) Q_{k-n}(x)}{H_{N-n}(x)(z-x)} dx \end{aligned} \tag{B.4}$$

where we used manipulations as in (2.36). It is important to keep in mind that we can not replace $P_N(x)$ by $\sqrt{F_{2n}(x)} H_{N-n}(x)$ as we did in (2.37). The reason in that the terms of order $\mathcal{O}(x^{-N})$ can not be dropped in the integral. To see this, note that $|z| > |x|$ for any x inside the contour of integration. Therefore, $1/(z-x)$ can be expanded as $(1/z)\sum_{l=0}^{\infty}(x/z)^l$.

A convenient way to deal with this is to write,

$$\oint_{z\ out} \frac{P_N(x)Q_{k-n}(x)}{H_{N-n}(x)(z-x)}dx = \oint_{z\ in} \frac{P_N(x)Q_{k-n}(x)}{H_{N-n}(x)(z-x)}dx - \oint_{C_z} \frac{P_N(x)Q_{k-n}(x)}{H_{N-n}(x)(z-x)}dx$$

where C_z is a small contour around z. "*out*" and "*in*" refer to the point $x = z$ being outside or inside the contour of integration. The first integral on the rhs gives $W'(z)$ as (2.36) indicates. The second one can be evaluated at the pole.

Using (2.24) to write,

$$H_{N-n}(z) = \frac{\sqrt{P_N^2(z) - 4\Lambda^{2N}}}{\sqrt{F_{2n}(z)}}.$$

and $y_m(z)$ from (2.39), we get (B.4) to be,

$$\mathrm{Tr}\,\frac{W'(\Phi_{cl})}{z-\Phi_{cl}} = \mathrm{Tr}\,\frac{1}{z-\Phi_{cl}}\left(W'(z) - y_m(z)\frac{P_N(z)}{\sqrt{P_N^2(z)-4\Lambda^{2N}}}\right) \tag{B.5}$$

The left hand side becomes a polynomial in z when combined with the term proportional to $W'(z)$ on the right hand side. The right hand side can also be simplified by using that $\mathrm{Tr}\,\frac{1}{z-\Phi_{cl}} = \frac{P_N'(z)}{P_N(z)}$. (B.5) is then,

$$\mathrm{Tr}\,\frac{W'(\Phi_{cl}) - W'(z)}{z-\Phi_{cl}} = y_m(z)\frac{P_N'(z)}{\sqrt{P_N^2(z)-4\Lambda^{2N}}} \tag{B.6}$$

In the right hand side we use the relation in the chiral ring of $\mathcal{N} = 2$

$$\frac{P_N'(z)}{\sqrt{P_N^2(z)-4\Lambda^{2N}}} = \left\langle \mathrm{Tr}\,\frac{1}{z-\Phi}\right\rangle.$$

For the left hand side we use the fact that

$$\mathrm{Tr}\,\frac{W'(\Phi_{cl}) - W'(z)}{z-\Phi_{cl}}$$

contains only $\mathrm{Tr}\,\Phi_{cl}^l$ with $l \leq k$ which for $k < 2N$ is equal to $\langle \mathrm{Tr}\,\Phi^l\rangle$,

$$\mathrm{Tr}\,\frac{W'(\Phi_{cl}) - W'(z)}{z-\Phi_{cl}} = \left\langle \mathrm{Tr}\,\frac{W'(\Phi) - W'(z)}{z-\Phi}\right\rangle.$$

Finally, we can write (B.6) as follows,

$$\left\langle \mathrm{Tr}\,\frac{W'(\Phi)}{z-\Phi}\right\rangle = \left\langle \mathrm{Tr}\,\frac{1}{z-\Phi}\right\rangle (W'(z) - y_m(z)) \tag{B.7}$$

By definition, the resolvent of the matrix model $R(z)$ satisfies $2R(z) = W'(z) - y_m(z)$. Therefore (B.7) becomes the generalized Konishi anomaly equation,

$$\left\langle \mathrm{Tr}\,\frac{W'(\Phi)}{z-\Phi}\right\rangle = 2R(z)\left\langle \mathrm{Tr}\,\frac{1}{z-\Phi}\right\rangle.$$

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C. Chebyshev polynomials

In this appendix we list the first six Chebyshev polynomials of first and second kind. These are used in the examples discussed in section 3.

The definition given in section 2.3 of Chebyshev polynomials of the first and second kind of degree t and $t-1$ respectively is the following. Set $x = \cos\theta$ and let

$$\mathcal{T}_t(x) = \cos(t\theta) \qquad \mathcal{U}_{t-1}(x) = \frac{1}{t}\frac{d\mathcal{T}_t}{dx}(x) = \frac{\sin(t\theta)}{\sin\theta}\,. \tag{C.1}$$

From this it is simple to compute the first of then,

t	$\mathcal{T}_t(x)$	$\mathcal{U}_t(x)$
1	x	$2x$
2	$2x^2-1$	$4x^2-1$
3	$4x^3-3x$	$8x^3-4x$
4	$8x^4-8x^2+1$	$16x^4-12x^2-1$
5	$16x^5-20x^3-5x$	$32x^5-32x^3+6x$
6	$32x^6-48x^4+18x^2-1$	$64x^6-80x^4+24x^2-1\,.$

D. A magnetic index

Here we want to explain an interesting result that was obtained in an unsuccessful attempt to find a new order parameter for these models. The attempt was unsuccessful because the object ν that we defined turned out to be computable in terms of objects that are already known.

We will examine closely the physical spectrum of particles in a theory in which classically $\mathrm{U}(N)$ is broken to a low energy subgroup such as $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$. (Eventually we will generalize to an arbitrary breaking pattern $\mathrm{U}(N) \to \prod_i \mathrm{U}(N_i)$.) Allowing for the strong quantum dynamics, the true low energy gauge group is $\mathrm{U}(1)\times\mathrm{U}(1)$. One $\mathrm{U}(1)$ is the center of $\mathrm{U}(N)$ and decouples from the theory. We will disregard it and focus on the $\mathrm{U}(1)$ that is not decoupled, whose generator we call Q.

Physical particles in the theory are in general electrically and magnetically charged with respect to this $\mathrm{U}(1)$. The product of the smallest nonzero electric charge q (of an unconfined magnetically neutral particle) times the smallest physical magnetic charge m is, according to Dirac, an integer ν. (We measure m as a multiple of the Dirac quantum $2\pi\hbar c$.) We will call this integer the magnetic index, and we claim that always

$$\nu t = N\,, \tag{D.1}$$

where t is the confinement index defined in section 2.1.

First let us verify this in examples. We consider first the case of $\mathrm{U}(4)$ broken to $\mathrm{U}(3)\times\mathrm{U}(1)$. We embed $\mathrm{U}(3)$ as the upper left 3×3 block in $\mathrm{U}(4)$. For the monopole, we consider an 't Hooft-Polyakov monopole embedded in $\mathrm{U}(4)$ in such a way that (in a

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unitary gauge) the magnetic charge is

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \tag{D.2}$$

where in other words the "1" acts in U(3) and the "−1" in U(1). To find the smallest electric charge that is not confined, we start with the massive W-bosons, which are a U(3) triplet embedded as follows in U(4):

$$\begin{pmatrix} 0 & & & W^1 \\ & 0 & & W^2 \\ & & 0 & W^3 \\ & & & 0 \end{pmatrix}. \tag{D.3}$$

The components W^1, W^2, and W^3 have, respectively, electric charges 1, 1, and 2, with respect to the electric charge generator given in (D.2).[12] The SU(3) singlet combination $\epsilon_{ijk}W^iW^jW^k$ is unconfined. Its electric charge is $q = 1 + 1 + 2 = 4$. Since the minimum magnetic charge m in this theory is 1 (in Dirac units), the product $\nu = qm$ equals 4. The breaking of U(4) to U(3) × U(1) gives a Coulomb branch, so $t = 1$, and $\nu t = 4$, as claimed in (D.1).

Now we consider the breaking of U(4) to U(2) × U(2). We embed the two factors of U(2) as the upper left and lower right blocks in U(4), and the monopole charge as

$$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}. \tag{D.4}$$

We focus on the upper right 2×2 block of W bosons, transforming as $(\mathbf{2}, \overline{\mathbf{2}})$ of U(2) × U(2):

$$\begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}. \tag{D.5}$$

The components $(W^{11}, W^{21}, W^{12}, W^{22})$ have (with respect to the charge generator in (D.4)) respectively electric charges $(1, 2, 0, 1)$, and the basic SU(2) × SU(2) singlet $\epsilon_{ij}\epsilon_{i'j'}W^{ii'} \times W^{jj'} = W^{11}W^{22} - W^{12}W^{21}$ hence has charge $q = 2$. In the confining branch, with $b = 0$ in the notation of section 2.1, the confining index is $t = 2$, the minimum magnetic charge is again $m = 1$, and $\nu = qm = 2$. So again $\nu t = 4$, as claimed.

What, however, happens in the branch with classical breaking to U(2) × U(2) and with $b = 1$? In this case, there is no confinement, and $t = 1$. What about ν? Shifting the value of b changes us from confinement to oblique confinement in one of the SU(2)'s, but this does not affect which combinations of W bosons are confined. So the minimum

[12]In evaluating the electric charge of an SU(3) singlet, we can simply use the charge matrix that appears in (D.2); subtracting an SU(3) generator to get a multiple of Q would not change the result.

electric charge, in the same units, is still $q = 2$. However, in section 2.1, we described a process of magnetic screening that occurs in this model when an external Wilson loop in the fundamental representation of SU(4) is considered; it is screened by the nucleation of an 't Hooft-Polyakov monopole with unit charge, and this is the mechanism that leads to $t = 1$. The fact that the 't Hooft-Polyakov monopole screens an external Wilson line means that, in the absence of the Wilson line, it would have infinite energy and would be confined (we explain the mechanism in more detail below). So the minimum unconfined magnetic charge when $b = 1$ actually has $m = 2$. Since q is still 2, this gives $\nu = 4$, which, with $t = 1$, is again consistent with $\nu t = 4$.

At least in hindsight, it should not come as too much of a surprise that we had to consider here the confinement of certain magnetic monopoles. We have taken account of electric confinement to evaluate q, so we should expect to consider magnetic confinement, that is confinement of some magnetic monopoles, to evaluate m.

More directly, the confinement for $b = 1$ of the minimum charge monopole arises as follows. The monopole has a collective coordinate whose quantization leads to the Julia-Zee dyon. There is only a single collective coordinate, and quantizing it gives the monopole equal color charges in the two SU(2)'s.[13] In SU(2) gauge theory, ordinary confinement means that a monopole is unconfined if its color electric charge is an integer multiple of the W boson color electric charge, and oblique confinement means that the color electric charge of an unconfined monopole is a half-integral multiple of the W boson color electric charge. Setting $b = 1$ means that one SU(2) has ordinary confinement and the other has oblique confinement. Given this, the states obtained by quantizing the collective coordinate are all confined, as we have claimed.

Proof of the relation. Now that we have given an idea of what the relation $\nu t = N$ means semiclassically, we proceed to a mathematical proof of this relation. In the process, we will also define the index ν for arbitrary rank of the low energy group; above, we assumed rank 1.

We start with the $\mathcal{N} = 2$ Coulomb branch. The low energy gauge group (omitting the center of $U(N)$) is $U(1)^{N-1}$. The massive gauge bosons and monopoles have charges that fill out a rank $2(N-1)$ lattice that we will call L. On L there is an integer-valued antisymmetric inner product that we will call ω. For example, if $N - 1 = 1$, there is a single U(1) and a vector in L is specified by giving electric and magnetic charges (e, g). Given two vectors $l = (e, g)$ and $l' = (e', g')$, the inner product is then $\omega(l, l') = eg' - e'g$. In general, for any number of U(1)'s, ω is defined by a similar formula, summing over all of the U(1)'s. The pfaffian of ω is in absolute value

$$|\mathrm{Pfaff}(\omega)| = N\,. \tag{D.6}$$

This relation reflects the fact that if we incorporate electric charges in the fundamental representation of $SU(N)$, then ω would become unimodular.

[13] Global SU(2) charge is not defined in the field of a magnetic monopole [22]. What we here call the color electric charge (for a given SU(2)) is the quantity that is well-defined. It originates from the projection of the charge generator in (D.4) into the SU(2) of interest; this charge is an integer for SU(2) generators, and would be a half-integer for external quarks.

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Now, after perturbing the theory by a superpotential (or in any other way, for that matter), some charges may condense. The condensed charges generate a sublattice M of L. The condensed objects are always "mutually local," in the sense that for $m, m' \in M$,

$$\omega(m, m') = 0\,. \tag{D.7}$$

A particle is confined if its charge vector l is such that $\omega(l, m) \neq 0$ for some $m \in M$. So the charge vectors of unconfined charges lie in $M^{\perp}$, the sublattice of L consisting of l such that $\omega(l, m) = 0$ for all $m \in M$. Physically, the charge of an unconfined particle can only be measured modulo M, since particles with quantum numbers in M have condensed. So the measureable charges in the low energy theory take values in $L' = M^{\perp}/M$. The antisymmetric product ω induces an analogous antisymmetric product ω' for the physical, unconfined charges in the low energy theory. The general definition of ν is

$$\nu = |\mathrm{Pfaff}(\omega')|\,. \tag{D.8}$$

ν determines how many extra types of charge could be added to the low energy theory (keeping fixed its gauge group) without any inconsistency.

L' is the lattice of charges that are not confined, and hence the quotient $L'' = L/L'$ might be regarded as the lattice of charges that are confined. There is a well-defined inner product $\omega'' : M \times L'' \to \mathbf{Z}$; for any $m \in M$, $l'' \in L''$, we simply lift l'' to an element $l \in L$ in an arbitrary fashion, and define $\omega''(m, l'') = \omega(m, l)$. The choice of l is well-defined modulo an element of $M^{\perp}$; adding such an element to l does not change $\omega(m, l)$. We can extend ω'' to an antisymmetric form on $M \oplus L''$ by defining $\omega''(l'', m) = -\omega(m, l'')$. The confinement index t is $t = \mathrm{Pfaff}(\omega'')$. In fact, the confinement index measures how many types of charge are conceivable (they have integer inner products with vectors in M) but are absent in the theory and can be represented instead by external Wilson loops.

Now it is a fact of linear algebra that in this situation

$$\mathrm{Pfaff}(\omega) = \mathrm{Pfaff}(\omega') \cdot \mathrm{Pfaff}(\omega''). \tag{D.9}$$

Putting all of this together, we get the promised relation $N = \nu t$.

If we are willing to make some unnatural choices, we can define the lattices and explain the key fact (D.9) in a way that might be easier to follow. There is not any natural way to embed L' as a sublattice of L, because vectors in L' can only be interpreted as elements of L modulo M. However, this indeterminacy is unimportant for us; we simply embed L' as a sublattice $L_1 \subset L$ by picking a basis of L' and lifting each basis vector to a vector in L, making a choice that (for each basis vector) is unique up to adding an element of M. If we do this, then ω' can be identified with ω_1, the restriction of ω to L_1.

Having identified L_1 as a sublattice of L, we can without further arbitrary choices define another sublattice $L_2 = L_1^{\perp}$; that is, L_2 consists of elements $l_2 \in L$ such that $\omega(l_1, l_2) = 0$ for all $l_1 \in L_1$. With respect to the decomposition $L = L_1 \oplus L_2$, ω is block-diagonal,

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}. \tag{D.10}$$

From this, (D.9) is clear.

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